RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A CLASSICAL PULSE BASE EXCITATION Revision A

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Introduction

Consider the single-degree-of-freedom system in Figure A-1.

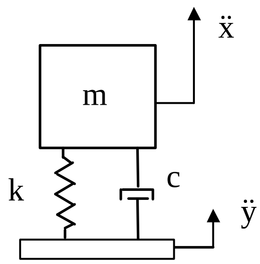

Figure A-1.

where

m is the mass,
c is the viscous damping coefficient,
k is the stiffness,
x is the absolute displacement of the mass,
y is the base input displacement.

A free-body diagram is shown in Figure A-2.

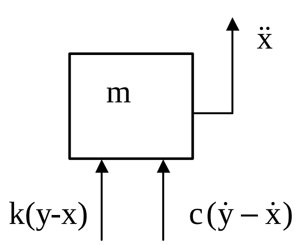

Figure A-2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{A-1}$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{A-2}$$

$$\text{Let } z = x - y \quad \text{(relative displacement)}$$
$$\dot{z} = \dot{x} - \dot{y}$$
$$\ddot{z} = \ddot{x} - \ddot{y}$$
$$\ddot{x} = \ddot{z} + \ddot{y}$$

Substituting the relative displacement terms into equation (A-2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \tag{A-3}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{A-4}$$

Dividing through by mass yields

$$\ddot{z} + (c/m)\dot{z} + (k/m)z = -\ddot{y} \tag{A-5}$$

By convention,

$$(c/m) = 2\xi\omega_n$$
$$(k/m) = {\omega_n}^2$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (A-5).

$$\ddot{z} + 2\xi\omega_n\dot{z} + {\omega_n}^2 z = -\ddot{y} \tag{A-6}$$

Half-sine Pulse

Consider the half-sine pulse given by equation (B-1).

$$\ddot{y}(t) = \begin{cases} A\sin\left(\frac{\pi t}{T}\right), & 0 \le t \le T \\ 0, & t > T \end{cases} \tag{B-1}$$

The equation of motion becomes

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -A\sin\left(\frac{\pi t}{T}\right), \qquad 0 \le t \le T \tag{B-2}$$

Let

$$\omega = \frac{\pi}{T} \tag{B-3}$$

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -A\sin(\omega t), \qquad 0 \le t \le T \tag{B-4}$$

Now take the Laplace transform.

$$L\left\{\ddot{z} + 2\xi\omega_n\,\dot{z} + \omega_n^2\,z\right\} = L\{-A\sin(\omega t)\} \tag{B-5}$$

$$\begin{aligned} & s^2 Z(s) - sz(0) - \dot{z}(0) \\ & + 2\xi\omega_n sZ(s) - 2\xi\omega_n z(0) \\ & + \omega_n{}^2 Z(s) = \frac{-A\,\omega}{s^2 + \omega^2} \end{aligned} \tag{B-6}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\}Z(s) + \{-1\}\dot{z}(0) + \{-s - 2\xi\omega_n\}z(0) = \frac{-A\,\omega}{s^2 + \omega^2} \tag{B-7}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\}Z(s) = \dot{z}(0) + \left\{s + 2\xi\omega_n\right\}z(0) - \frac{A\,\omega}{s^2 + \omega^2} \tag{B-8}$$

$$Z(s) = \left\{\frac{\dot{z}(0) + \left\{s + 2\xi\omega_n\right\}z(0)}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} - \left\{\frac{A\,\omega}{s^2 + \omega^2}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{B-9}$$

Let

$$Z(s) = Z_n(s) + Z_f(s) \tag{B-10}$$

where

$$Z_n(s) = \left\{\frac{\dot{z}(0) + \left\{s + 2\xi\omega_n\right\}z(0)}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{B-11}$$

$$Z_f(s) = -\left\{\frac{A\,\omega}{s^2 + \omega^2}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{B-12}$$

Consider the denominator term,

$$s^2 + 2\,\xi\,\omega_n s + \omega_n{}^2 = \left(s + \xi\,\omega_n\right)^2 + \omega_n{}^2 - \left(\xi\,\omega_n\right)^2 \tag{B-13}$$

$$s^2 + 2\,\xi\,\omega_n s + \omega_n{}^2 = \left(s + \xi\,\omega_n\right)^2 + \omega_n{}^2\left(1 - \xi^2\right) \tag{B-14}$$

Now define the damped natural frequency,

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \tag{B-15}$$

Substitute equation (B-15) into (B-14),

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = \left(s + \xi\omega_n\right)^2 + \omega_d{}^2 \tag{B-16}$$

Substitute equation (B-16) into (B-12).

$$Z_n(s) = \left\{\frac{\dot{z}(0) + \left\{s + 2\xi\omega_n\right\}z(0)}{\left(s + \xi\omega_n\right)^2 + \omega_d{}^2}\right\} \tag{B-17}$$

Rearrange the terms into a convenient format prior to the inverse Laplace transform.

$$Z_n(s) = \left\{ \frac{(s+\xi\omega_n)z(0)}{(s+\xi\omega_n)^2 + \omega_d{}^2} \right\} + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{(s+\xi\omega_n)^2 + \omega_d{}^2} \right\} \qquad \text{(B-18)}$$

$$Z_n(s) = \left\{ \frac{(s+\xi\omega_n)z(0)}{(s+\xi\omega_n)^2 + \omega_d{}^2} \right\} + \left\{ \frac{\left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\}\omega_d}{(s+\xi\omega_n)^2 + \omega_d{}^2} \right\} \qquad \text{(B-19)}$$

Take the inverse Laplace transform using Reference 1.

$$z_n(t) = z(0)\exp(-\xi\omega_n t)\cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\}\exp(-\xi\omega_n t)\sin(\omega_d t) \qquad \text{(B-20)}$$

$$z_n(t) = \exp(-\xi\omega_n t)\left\{ z(0)\cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\}\sin(\omega_d t) \right\} \qquad \text{(B-21)}$$

Take the first derivative to determine the relative velocity.

$$\dot{z}_n(t) = -\xi\omega_n \exp(-\xi\omega_n t)\left\{ z(0)\cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\}\sin(\omega_d t) \right\}$$
$$+ \exp(-\xi\omega_n t)\{-\omega_d z(0)\sin(\omega_d t) + \{\dot{z}(0) + (\xi\omega_n)z(0)\}\cos(\omega_d t)\}$$

(B-22)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{ -\xi\omega_n z(0)\cos(\omega_d t) - \xi\omega_n \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\}\sin(\omega_d t) \right\}$$
$$+ \exp(-\xi\omega_n t)\{-\omega_d z(0)\sin(\omega_d t) + \{\dot{z}(0) + (\xi\omega_n)z(0)\}\cos(\omega_d t)\}$$

(B-23)

$$\dot{z}_n(t) =$$

$$\exp(-\xi\omega_n t)\{\{-\xi\omega_n z(0) + \dot{z}(0) + (\xi\omega_n)z(0)\}\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\left\{-\omega_d z(0) - \xi\omega_n\left[\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right]\right\}\sin(\omega_d t)\right\}$$

(B-24)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_d{}^2 z(0) - \xi\omega_n[\dot{z}(0) + (\xi\omega_n)z(0)]\right\}\sin(\omega_d t)\right\}$$

(B-25)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_d{}^2 z(0) - \xi\omega_n\dot{z}(0) - (\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(B-26)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_n{}^2\left(1-\xi^2\right)z(0) - \xi\omega_n\dot{z}(0) - (\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(B-27)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{\left(-\omega_n{}^2 + \xi^2\omega_n{}^2\right)z(0) - \xi\omega_n\dot{z}(0) - (\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(B-28)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_n{}^2 z(0) - \xi\omega_n \dot{z}(0)\right\}\sin(\omega_d t)\right\}$$

(B-29)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t) + \frac{1}{\omega_d}\left\{-\omega_n{}^2 z(0) - \xi\omega_n \dot{z}(0)\right\}\sin(\omega_d t)\right\}$$

(B-30)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\{-\omega_n z(0) - \xi\,\dot{z}(0)\}\sin(\omega_d t)\right\}$$

(B-31)

Take the second derivative to determine the acceleration.

$$\ddot{z}_n(t) = -\xi\omega_n\exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\{-\omega_n z(0) - \xi\,\dot{z}(0)\}\sin(\omega_d t)\right\}$$
$$+\exp(-\xi\omega_n t)\{-\omega_d \dot{z}(0)\sin(\omega_d t) + \omega_n\{-\omega_n z(0) - \xi\,\dot{z}(0)\}\cos(\omega_d t)\}$$

(B-32)

$$\ddot{z}_n(t) = \exp(-\xi\omega_n t)\left\{-\xi\omega_n \dot{z}(0)\cos(\omega_d t) - \frac{\xi\omega_n{}^2}{\omega_d}\{-\omega_n z(0) - \xi\,\dot{z}(0)\}\sin(\omega_d t)\right\}$$
$$+\exp(-\xi\omega_n t)\{-\omega_d \dot{z}(0)\sin(\omega_d t) + \omega_n\{-\omega_n z(0) - \xi\,\dot{z}(0)\}\cos(\omega_d t)\}$$

(B-33)

$$\ddot{z}_n(t) = \exp(-\xi\omega_n t)\{-\xi\omega_n \dot{z}(0) + \omega_n\{-\omega_n z(0) - \xi\dot{z}(0)\}\}\cos(\omega_d t)$$
$$+\exp(-\xi\omega_n t)\left\{-\omega_d \dot{z}(0) - \frac{\xi\omega_n{}^2}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\right\}\sin(\omega_d t)$$

(B-34)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_d{}^2\dot{z}(0) - \xi\omega_n{}^2\{-\omega_n z(0) - \xi\dot{z}(0)\}\right\}\sin(\omega_d t)$$

(B-35)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_d{}^2\dot{z}(0) + \xi\omega_n{}^3 z(0) + \xi^2\omega_n{}^2\dot{z}(0)\right\}\sin(\omega_d t)$$

(B-36)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_n{}^2\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n{}^3 z(0) + \xi^2\omega_n{}^2\dot{z}(0)\right\}\sin(\omega_d t)$$

(B-37)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{-\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n z(0) + \xi^2\dot{z}(0)\right\}\sin(\omega_d t)$$

(B-38)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{\xi\omega_n z(0) + \left(-1+2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)$$

(B-39)

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\dot{z}(0)\}\cos(\omega_d t)$$
$$-\frac{\omega_n^{\,2}}{\omega_d}\exp(-\xi\omega_n t)\left\{-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)$$

(B-40)

$$\ddot{z}_n(t) =$$
$$-\exp(-\xi\omega_n t)\left\{\omega_n\left[\omega_n z(0) + 2\xi\dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n^{\,2}}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\}$$

(B-41)

$$\ddot{z}_n(t) =$$
$$-\omega_n \exp(-\xi\omega_n t)\left\{\left[\omega_n z(0) + 2\xi\dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\}$$

(B-42)

Recall equation (B-12).

$$Z_f(s) = -\left\{\frac{A\,\omega}{s^2+\omega^2}\right\}\left\{\frac{1}{s^2+2\xi\omega_n s+\omega_n^{\,2}}\right\} \tag{B-43}$$

Expand into partial fractions using Reference 2.

$$\left\{\frac{1}{s^2+\omega^2}\right\}\left\{\frac{A\omega}{s^2+2\xi\omega_n s+\omega_n^2}\right\} =$$

$$+A\omega\frac{\left[-2\xi\omega_n\right]s+\left[-\left(\omega^2-\omega_n^2\right)\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\,\omega\omega_n\right)^2\right]\left[s^2+\omega^2\right]}$$

$$+A\omega\frac{\left[2\xi\omega_n\right]s+\left[\left(\omega^2-\omega_n^2\right)+\left(2\xi\omega_n\right)^2\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\,\omega\omega_n\right)^2\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]}$$

(B-44)

$$Z_f(s) =$$

$$-A\omega\frac{\left[-2\xi\omega_n\right]s+\left[-\left(\omega^2-\omega_n^2\right)\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\,\omega\omega_n\right)^2\right]\left[s^2+\omega^2\right]}$$

$$-A\omega\frac{\left[2\xi\omega_n\right]s+\left[\left(\omega^2-\omega_n^2\right)+\left(2\xi\omega_n\right)^2\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+\left(2\xi\,\omega\omega_n\right)^2\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]}$$

(B-45)

$$Z_f(s) =$$

$$-\frac{-2A\xi\omega\omega_n}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[\frac{s+\left(\frac{\omega^2-\omega_n^2}{2\xi\omega_n}\right)}{s^2+\omega^2}\right]$$

$$-\frac{2A\xi\omega\omega_n}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[\frac{s+\left(\frac{\left(\omega^2-\omega_n^2\right)+(2\xi\omega_n)^2}{2\xi\omega_n}\right)}{s^2+2\xi\omega_n s+\omega_n^2}\right]$$

(B-46)

$$Z_f(s) =$$

$$+\frac{2A\xi\omega\omega_n}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[\frac{s+\left(\frac{\omega^2-\omega_n^2}{2\xi\omega_n}\right)}{s^2+\omega^2}\right]$$

$$-\frac{2A\xi\omega\omega_n}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[\frac{s+\left(\frac{\left(\omega^2-\omega_n^2\right)+(2\xi\omega_n)^2}{2\xi\omega_n}\right)}{s^2+2\xi\omega_n s+\omega_n^2}\right]$$

(B-47)

Take the inverse Laplace transform using Reference 1.

$$z_f(t) =$$

$$+\frac{2A\xi\omega\omega_n}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[\cos(\omega t)+\left(\frac{\omega^2-\omega_n^2}{2\xi\omega\omega_n}\right)\sin(\omega t)\right]$$

$$-\frac{2A\xi\omega\omega_n[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[\cos(\omega_d t)+\left(\frac{\left(\frac{\left(\omega^2-\omega_n^2\right)+(2\xi\omega_n)^2}{2\xi\omega_n}\right)-\xi\omega_n}{\omega_d}\right)\sin(\omega_d t)\right]$$

(B-48)

$$z_f(t) =$$

$$+\frac{A}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[(2\xi\omega\omega_n)\cos(\omega t)+\left(\omega^2-\omega_n^2\right)\sin(\omega t)\right]$$

$$-\frac{2A\xi\omega\omega_n[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[\cos(\omega_d t)+\left(\frac{\left(\omega^2-\omega_n^2\right)+(2\xi\omega_n)^2-2(\xi\omega_n)^2}{2\xi\omega_n\omega_d}\right)\sin(\omega_d t)\right]$$

(B-49)

$$z_f(t) =$$

$$+ \frac{A}{\left[\left(\omega^2 - \omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2\right]}\left[\left(2\xi\omega\omega_n\right)\cos(\omega t) + \left(\omega^2 - \omega_n^2\right)\sin(\omega t)\right]$$

$$- \frac{2A\xi\omega\omega_n\left[\exp(-\xi\omega_n t)\right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2\right]}\left[\cos(\omega_d t) + \left(\frac{\left(\omega^2 - \omega_n^2\right) + 2\left(\xi\omega_n\right)^2}{2\xi\omega_n\omega_d}\right)\sin(\omega_d t)\right]$$

(B-50)

$$z_f(t) =$$

$$+ \frac{A}{\left[\left(\omega^2 - \omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2\right]}\left[\left(2\xi\omega\omega_n\right)\cos(\omega t) + \left(\omega^2 - \omega_n^2\right)\sin(\omega t)\right]$$

$$- \frac{\frac{2A\xi\omega\omega_n}{2\xi\omega_n\omega_d}\left[\exp(-\xi\omega_n t)\right]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + \left(2\xi\omega\omega_n\right)^2\right]}\left[\left(2\xi\omega_n\omega_d\right)\cos(\omega_d t) + \left(\left(\omega^2 - \omega_n^2\right) + 2\left(\xi\omega_n\right)^2\right)\sin(\omega_d t)\right]$$

(B-51)

The relative displacement is thus

$$z_f(t) =$$

$$+ \frac{A}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[(2\xi\omega\omega_n)\cos(\omega t) + \left(\omega^2 - \omega_n^2\right)\sin(\omega t)\right]$$

$$- \frac{\frac{A\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[(2\xi\omega_n\omega_d)\cos(\omega_d t) + \left(\omega^2 - \omega_n^2\left(1 - 2\xi^2\right)\right)\sin(\omega_d t)\right]$$

(B-52)

Take the first derivative to determine the relative velocity.

$$\dot{z}_f(t) =$$

$$+ \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2\right)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[\frac{\xi\omega_n}{\omega_d}(2\xi\omega_n\omega_d) - \left(\omega^2 - \omega_n^2\left(1 - 2\xi^2\right)\right)\right]\cos(\omega_d t)$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[(2\xi\omega_n\omega_d) + \frac{\xi\omega_n}{\omega_d}\left(\omega^2 - \omega_n^2\left(1 - 2\xi^2\right)\right)\right]\sin(\omega_d t)$$

(B-53)

$$\dot{z}_f(t) =$$

$$+ \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2\right)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[\left(2\xi^2\omega_n^2\right) - \left(\omega^2 - \omega_n^2\left(1 - 2\xi^2\right)\right)\right]\cos(\omega_d t)$$

$$+ \frac{\frac{A\xi\omega_n\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[\left(2\omega_d^2\right) + \left(\omega^2 - \omega_n^2\left(1 - 2\xi^2\right)\right)\right]\sin(\omega_d t)$$

(B-54)

$$\dot{z}_f(t) =$$

$$+ \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2\right)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[2\xi^2\omega_n^2 - \omega^2 + \omega_n^2\left(1 - 2\xi^2\right)\right]\cos(\omega_d t)$$

$$+ \frac{\frac{A\xi\omega_n\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[2\omega_n^2\left(1 - \xi^2\right) + \omega^2 - \omega_n^2\left(1 - 2\xi^2\right)\right]\sin(\omega_d t)$$

(B-55)

$$\dot{z}_f(t) =$$

$$+ \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2\right)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[\omega_n^2 - \omega^2\right]\cos(\omega_d t)$$

$$+ \frac{\frac{A\xi\omega_n\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[2\omega_n^2 - 2\xi^2\omega_n^2 + \omega^2 - \omega_n^2 + 2\xi^2\omega_n^2\right]\sin(\omega_d t)$$

(B-56)

$$\dot{z}_f(t) =$$

$$+ \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-(2\xi\omega\omega_n)\sin(\omega t) + \left(\omega^2 - \omega_n^2\right)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[\omega_n^2 - \omega^2\right]\cos(\omega_d t)$$

$$+ \frac{\frac{A\xi\omega_n\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[\omega_n^2 + \omega^2\right]\sin(\omega_d t)$$

(B-57)

$$\dot{z}_f(t) =$$

$$+ \frac{A\omega}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[\left(\omega^2-\omega_n^2\right)\cos(\omega t)-(2\xi\omega\omega_n)\sin(\omega t)\right]$$

$$+\frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\left[\omega_n^2-\omega^2\right]\cos(\omega_d t)+\frac{\xi\omega_n}{\omega_d}\left[\omega_n^2+\omega^2\right]\sin(\omega_d t)\right\}$$

(B-58)

Take the second derivative to determine the relative acceleration.

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t)-(2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+\frac{-A\xi\omega_n\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\left[\omega_n^2-\omega^2\right]\cos(\omega_d t)+\frac{\xi\omega_n}{\omega_d}\left[\omega_n^2+\omega^2\right]\sin(\omega_d t)\right\}$$

$$+\frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{-\omega_d\left[\omega_n^2-\omega^2\right]\sin(\omega_d t)+\xi\omega_n\left[\omega_n^2+\omega^2\right]\cos(\omega_d t)\right\}$$

(B-59)

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2 - \omega_n^2\right)\sin(\omega t) - (2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{-\xi\omega_n\left[\omega_n^2 - \omega^2\right]\cos(\omega_d t) - \frac{\xi^2\omega_n^{\ 2}}{\omega_d}\left[\omega_n^2 + \omega^2\right]\sin(\omega_d t)\right\}$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{-\omega_d\left[\omega_n^2 - \omega^2\right]\sin(\omega_d t) + \xi\omega_n\left[\omega_n^2 + \omega^2\right]\cos(\omega_d t)\right\}$$

(B-60)

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2 - \omega_n^2\right)\sin(\omega t) - (2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{-\xi\omega_n\left[\omega_n^2 - \omega^2\right] + \xi\omega_n\left[\omega_n^2 + \omega^2\right]\right\}\cos(\omega_d t)$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{-\frac{\xi^2\omega_n^{\ 2}}{\omega_d}\left[\omega_n^2 + \omega^2\right] - \omega_d\left[\omega_n^2 - \omega^2\right]\right\}\sin(\omega_d t)$$

(B-61)

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2 - \omega_n^2\right)\sin(\omega t) - (2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{2\xi\omega_n\omega^2\right\}\cos(\omega_d t)$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{\frac{1}{\omega_d}\right\}\left\{-\xi^2\omega_n{}^2\left[\omega_n^2 + \omega^2\right] - \omega_d{}^2\left[\omega_n^2 - \omega^2\right]\right\}\sin(\omega_d t)$$

(B-62)

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2 - \omega_n^2\right)\sin(\omega t) - (2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{2\xi\omega_n\omega^2\right\}\cos(\omega_d t)$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{\frac{1}{\omega_d}\right\}\left\{-\xi^2\omega_n{}^4 - \xi^2\omega_n{}^2\omega^2 - \omega_d{}^2\omega_n^2 + \omega_d{}^2\omega^2\right\}\sin(\omega_d t)$$

(B-63)

$$\dot{z}_f(t) =$$

$$+ \frac{A\omega}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-(2\xi\omega\omega_n)\sin(\omega t)+\left(\omega^2-\omega_n^2\right)\cos(\omega t)\right]$$

$$+ \frac{\frac{A\xi\omega_n\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[(2\xi\omega_n\omega_d)\cos(\omega_d t)+\left(\omega^2-\omega_n^2\left(1-2\xi^2\right)\right)\sin(\omega_d t)\right]$$

$$- \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-(2\xi\omega_n\omega_d)\sin(\omega_d t)+\left(\omega^2-\omega_n^2\left(1-2\xi^2\right)\right)\cos(\omega_d t)\right]$$

(B-64)

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t)-(2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{2\xi\omega_n\omega^2\right\}\cos(\omega_d t)$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\frac{1}{\omega_d}\right\}\left\{-\xi^2\omega_n{}^4-\xi^2\omega_n{}^2\omega^2-\omega_n{}^4\left(1-\xi^2\right)+\omega^2\omega_n{}^2\left(1-\xi^2\right)\right\}\sin(\omega_d t)$$

(B-65)

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t)-(2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+\frac{A\omega\left[\exp(-\xi\omega_n t)\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{2\xi\omega_n\omega^2\right\}\cos(\omega_d t)$$

$$+\frac{A\omega\left[\exp(-\xi\omega_n t)\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\frac{\omega_n{}^2}{\omega_d}\right\}\left\{-\xi^2\omega_n{}^2-\xi^2\omega^2-\omega_n{}^2\left(1-\xi^2\right)+\omega^2\left(1-\xi^2\right)\right\}\sin(\omega_d t)$$

(B-66)

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t)-(2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+\frac{A\omega\left[\exp(-\xi\omega_n t)\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{2\xi\omega_n\omega^2\right\}\cos(\omega_d t)$$

$$+\frac{A\omega\left[\exp(-\xi\omega_n t)\right]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\frac{\omega_n{}^2}{\omega_d}\right\}\left\{-\xi^2\omega_n{}^2-\xi^2\omega^2-\omega_n{}^2+\xi^2\omega_n{}^2+\omega^2-\xi^2\omega^2\right\}\sin(\omega_d t)$$

(B-67)

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t)-(2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+\frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{2\xi\omega_n\omega^2\right\}\cos(\omega_d t)$$

$$+\frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\frac{\omega_n{}^2}{\omega_d}\right\}\left\{-2\xi^2\omega^2-\omega_n{}^2+\omega^2\right\}\sin(\omega_d t)$$

(B-68)

$$\ddot{z}_f(t) =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t)-(2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+\frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{2\xi\omega_n\omega^2\right\}\cos(\omega_d t)$$

$$+\frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\frac{\omega_n{}^2}{\omega_d}\right\}\left\{-\omega_n{}^2+\omega^2\left(1-2\xi^2\right)\right\}\sin(\omega_d t)$$

(B-69)

$$\ddot{z}_f(t) \quad =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t)-(2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+\frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\left\{2\xi\omega_n\omega^2\right\}\cos(\omega_d t)+\left\{\frac{\omega_n{}^2}{\omega_d}\right\}\left\{-\omega_n{}^2+\omega^2\left(1-2\xi^2\right)\right\}\sin(\omega_d t)\right\}$$

(B-70)

$$\ddot{z}_f(t) \quad =$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t)-(2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+\frac{A\omega\omega_n[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2+(2\xi\omega\omega_n)^2\right]}\left\{\left\{2\xi\omega^2\right\}\cos(\omega_d t)+\left\{\frac{\omega_n}{\omega_d}\right\}\left\{-\omega_n{}^2+\omega^2\left(1-2\xi^2\right)\right\}\sin(\omega_d t)\right\}$$

(B-71)

The total relative displacement for $0 \le t \le T$ is

$$z(t) = z_n(t) + z_f(t) \tag{B-72}$$

Substitute equations (B-21) and (B-53) into (B-73).

$$z(t) = \exp(-\xi\omega_n t)\left\{ z(0)\cos(\omega_d t) + \left\{ \frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d} \right\} \sin(\omega_d t) \right\}$$

$$+ \frac{A}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]} \left[(2\xi\omega\omega_n)\cos(\omega t) + \left(\omega^2 - \omega_n^2\right)\sin(\omega t)\right]$$

$$- \frac{\frac{A\omega}{\omega_d}[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]} \left[(2\xi\omega_n\omega_d)\cos(\omega_d t) + \left(\omega^2 - \omega_n^2\left(1 - 2\xi^2\right)\right)\sin(\omega_d t)\right]$$

(B-73)

The total relative velocity for $0 \le t \le T$ is

$$\dot{z}(t) = \dot{z}_n(t) + \dot{z}_f(t) \quad \text{(B-74)}$$

$$\dot{z}(t) = \exp(-\xi\omega_n t)\left\{ \dot{z}(0)\ \cos(\omega_d t) + \frac{\omega_n}{\omega_d}\{-\omega_n z(0) - \xi\,\dot{z}(0)\}\sin(\omega_d t) \right\}$$

$$+ \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]} \left[\left(\omega^2 - \omega_n^2\right)\cos(\omega t) - (2\xi\omega\omega_n)\sin(\omega t)\right]$$

$$+ \frac{A\omega[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]} \left\{ \left[\omega_n^2 - \omega^2\right]\cos(\omega_d t) + \frac{\xi\omega_n}{\omega_d}\left[\omega_n^2 + \omega^2\right]\sin(\omega_d t) \right\}$$

(B-75)

The total relative acceleration for $0 \leq t \leq T$ is

$$\ddot{z}(t) = \ddot{z}_n(t) + \ddot{z}_f(t) \tag{B-76}$$

$$\begin{aligned}
\ddot{z}(t) = &-\omega_n \exp(-\xi\omega_n t)\left\{[\omega_n z(0) + 2\xi\dot{z}(0)]\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\} \\
&+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t) - (2\xi\omega\omega_n)\cos(\omega t)\right] \\
&+ \frac{A\omega\omega_n[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{\left\{2\xi\omega^2\right\}\cos(\omega_d t) + \left\{\frac{\omega_n}{\omega_d}\right\}\left\{-\omega_n^2 + \omega^2\left(1-2\xi^2\right)\right\}\sin(\omega_d t)\right\}
\end{aligned} \tag{B-77}$$

The total absolute acceleration for $0 \leq t \leq T$ is

$$\ddot{x}(t) = \ddot{z}_n(t) + \ddot{z}_f(t) + \ddot{y}(t) \tag{B-78}$$

$$\ddot{x}(t) = -\omega_n \exp(-\xi\omega_n t)\left\{\left[\omega_n z(0) + 2\xi\dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\}$$

$$+ \frac{A\omega^2}{\left[\left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[-\left(\omega^2-\omega_n^2\right)\sin(\omega t) - (2\xi\omega\omega_n)\cos(\omega t)\right]$$

$$+ \frac{A\omega\omega_n[\exp(-\xi\omega_n t)]}{\left[\left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left\{\left\{2\xi\omega^2\right\}\cos(\omega_d t) + \left\{\frac{\omega_n}{\omega_d}\right\}\left\{-\omega_n{}^2 + \omega^2\left(1-2\xi^2\right)\right\}\sin(\omega_d t)\right\}$$

$$+ A\sin(\omega t)$$

(B-79)

The relative displacement at t = T is

$$z(T) = \exp(-\xi\omega_n T)\left\{z(0)\cos(\omega_d T) + \left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d T)\right\}$$

$$+ \frac{A}{\left[\left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[(2\xi\omega\omega_n)\cos(\omega T) + \left(\omega^2-\omega_n^2\right)\sin(\omega T)\right]$$

$$- \frac{\frac{A\omega}{\omega_d}[\exp(-\xi\omega_n T)]}{\left[\left(\omega^2-\omega_n^2\right)^2 + (2\xi\omega\omega_n)^2\right]}\left[(2\xi\omega_n\omega_d)\cos(\omega_d T) + \left(\omega^2-\omega_n^2\left(1-2\xi^2\right)\right)\sin(\omega_d T)\right]$$

(B-80)

The relative velocity at t = T is

$$\dot{z}(T) = \exp(-\xi\omega_n T)\left\{\dot{z}(0)\ \cos(\omega_d T) + \frac{\omega_n}{\omega_d}\{-\omega_n z(0) - \xi\,\dot{z}(0)\}\sin(\omega_d T)\right\}$$

$$+\ \frac{A\omega}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2\right]}\left[\left(\omega^2 - \omega_n^2\right)\cos(\omega T) - (2\xi\omega\omega_n)\sin(\omega T)\right]$$

$$+\frac{A\omega[\exp(-\xi\omega_n T)]}{\left[\left(\omega^2 - \omega_n^2\right)^2 + (2\xi\,\omega\,\omega_n)^2\right]}\left\{\left[\omega_n^2 - \omega^2\right]\cos(\omega_d T) + \frac{\xi\,\omega_n}{\omega_d}\left[\omega_n^2 + \omega^2\right]\sin(\omega_d T)\right\}$$

(B-81)

The relative displacement for t > T is found by adding a delay into equation (B-21).

$$z(t) = \exp(-\xi\omega_n(t-T))\left\{z(T)\cos(\omega_d(t-T)) + \left\{\frac{\dot{z}(T) + (\xi\omega_n)z(T)}{\omega_d}\right\}\sin(\omega_d(t-T))\right\}$$

(B-82)

Note that the absolute displacement is equal to the relative displacement for t > T.

The relative velocity for t > T is

$$\dot{z}(t) = \exp(-\xi\omega_n(t-T))\left\{\dot{z}(T)\ \cos(\omega_d(t-T)) + \frac{\omega_n}{\omega_d}\{-\omega_n z(T) - \xi\,\dot{z}(T)\}\sin(\omega_d(t-T))\right\}$$

(B-83)

Note that the absolute velocity is equal to the relative velocity for t > T.

The relative acceleration for t > T is

$$\ddot{z}(t) =$$

$$-\omega_n \exp(-\xi\omega_n(t-T))\{[\omega_n z(T) + 2\xi\dot{z}(T)]\cos(\omega_d(t-T))\}$$

$$-\omega_n \exp(-\xi\omega_n(t-T))\left\{\frac{\omega_n}{\omega_d}\left[-\xi\omega_n z(T) + \left(1-2\xi^2\right)\dot{z}(T)\right]\sin(\omega_d(t-T))\right\}$$

(B-84)

Note that the absolute acceleration is equal to the relative acceleration for t > T.

$$\ddot{x}(t) =$$

$$-\omega_n \exp(-\xi\omega_n(t-T))\{[\omega_n z(T) + 2\xi\dot{z}(T)]\cos(\omega_d(t-T))\}$$

$$-\omega_n \exp(-\xi\omega_n(t-T))\left\{\frac{\omega_n}{\omega_d}\left[-\xi\omega_n z(T) + \left(1-2\xi^2\right)\dot{z}(T)\right]\sin(\omega_d(t-T))\right\}$$

(B-85)

Example

An example is shown in Figure B-1. The input is an 10 G, 11 millisecond half-sine shock.

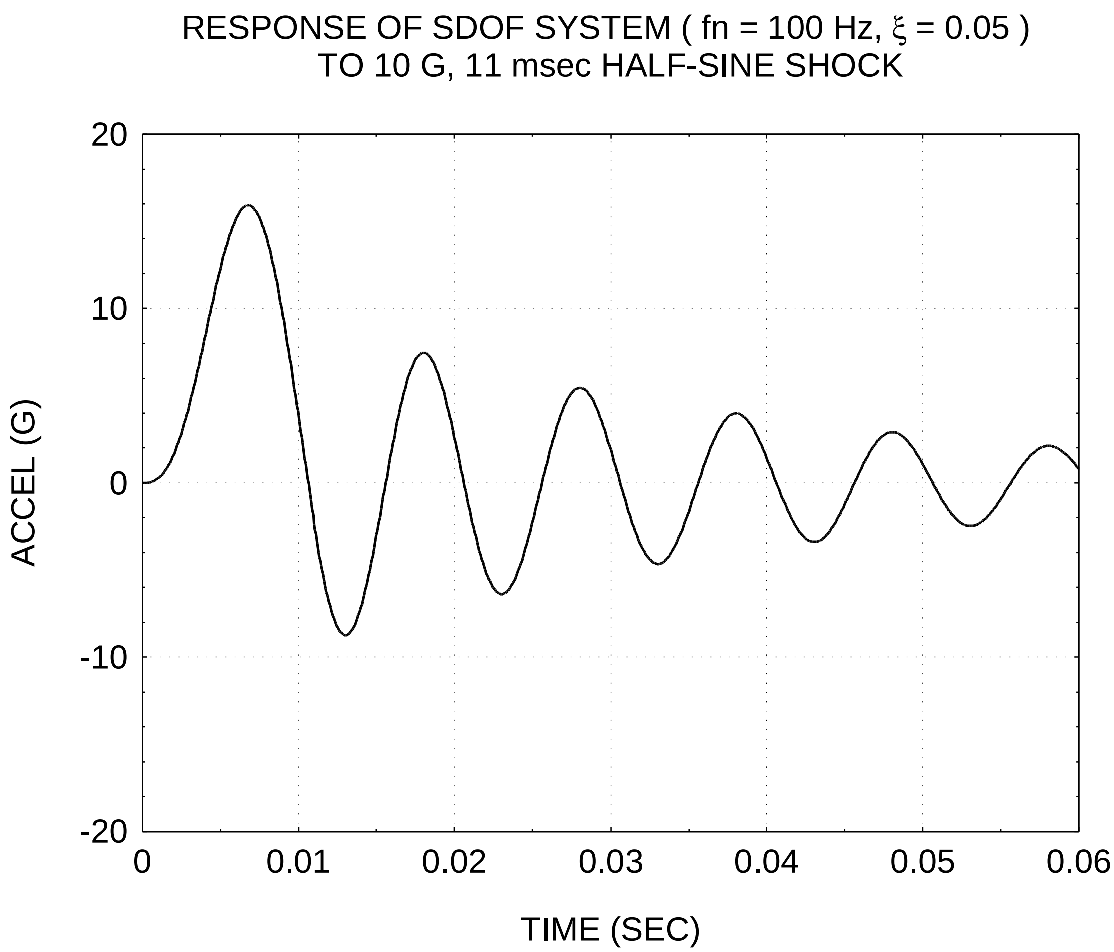

Figure B-1.

References

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2. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Vibrationdata.com Publications, 1999.