PENDULUM OSCILLATION, Revision C

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Background

Galileo Galilei lived from 1564 to 1642.

Galileo entered the University of Pisa in 1581 to study medicine. According to legend, he observed a lamp swinging back and forth in the Pisa cathedral. He noticed that the period of time required for one oscillation was the same, regardless of the distance of travel. This distance is called amplitude.

Later, Galileo performed experiments to verify his observation. He also suggested that this principle could be applied to the regulation of clocks.

Simple Pendulum

Consider a simple pendulum as shown in Figure A-1. A free-body diagram of the mass is shown in Figure A-2.

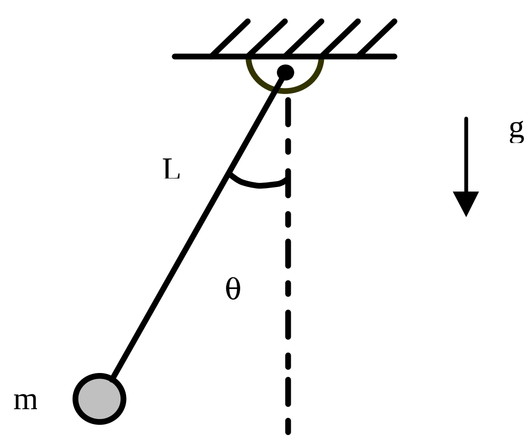
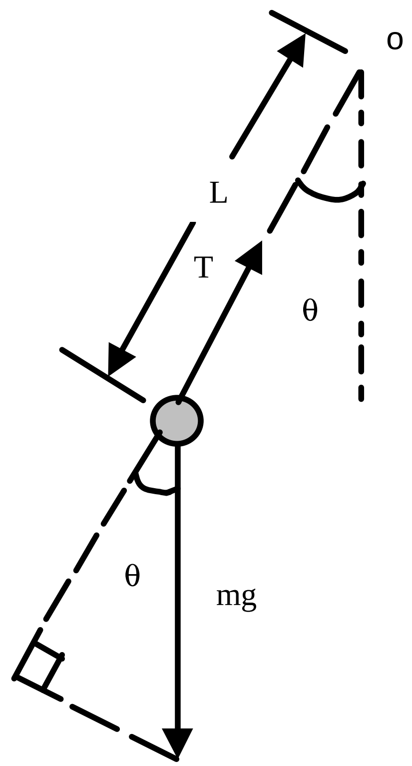


Figure A-1. Simple Pendulum



T = tension in the connecting rod or cable.
m = mass.
g = gravitational acceleration.
L = length from pivot to center of mass.
θ = angular displacement

Figure A-2. Free-body Diagram

Sum the moments about the pivot point o. Let clockwise be positive

$$\sum Mo = mL^2\ddot{\theta} \tag{A-1}$$

$$mL^2\ddot{\theta} = -mgL\sin\theta \qquad \text{(A-2)}$$

Divide through by mL.

$$L\ddot{\theta} = -g\sin\theta \qquad \text{(A-3)}$$

$$L\ddot{\theta} + g\sin\theta = 0 \qquad \text{(A-4)}$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \qquad \text{(A-5)}$$

Now assume small angular displacements, such that

$$\sin\theta \approx \theta \qquad \text{(A-6)}$$

By substitution, a linear equation is obtained.

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \qquad \text{(A-7)}$$

Now assume that the free oscillation solution is

$$\theta = A\sin\omega_n t + B\cos\omega_n t \qquad \text{(A-8)}$$

where

ω_n is the natural frequency,

A and B are coefficients which depend on the initial conditions.

The angular velocity is

$$\dot{\theta} = \omega_n\left\{A\cos\omega_n t - B\sin\omega_n t\right\} \qquad \text{(A-9)}$$

The angular acceleration is

$$\ddot{\theta} = -\omega_n{}^2\left\{A\sin\omega_n t + B\cos\omega_n t\right\} \qquad \text{(A-10)}$$

Substitute equations (A-8) and (A-10) into (A-7).

$$-\omega_n^{\,2}\left\{A\sin\omega_n t+B\cos\omega_n t\right\}+\frac{g}{L}\left\{A\sin\omega_n t+B\cos\omega_n t\right\}=0 \tag{A-11}$$

$$-\omega_n^{\,2}+\frac{g}{L}=0 \tag{A-12}$$

$$\omega_n^{\,2}=\frac{g}{L} \tag{A-13}$$

The natural frequency is thus

$$\omega_n=\sqrt{\frac{g}{L}} \tag{A-14}$$

Note that ω_n has dimensions of radians/time. The typical unit is radians/second.

The natural frequency can be expressed as

$$f_n = \frac{\omega_n}{2\pi} \tag{A-15}$$

where

> f_n has dimensions of cycles/time. The typical unit is cycles/second, which is also called Hertz.

The period T is related to the natural frequency by

$$T = \frac{1}{f_n} \tag{A-16}$$

The period is the time required for one complete cycle of oscillation.

<u>Inverted Pendulum with Rotational Spring at Pivot</u>

Consider the inverted pendulum with the rotational spring shown in Figure B-1. The forces and moments are shown in the diagram is in Figure B-2.

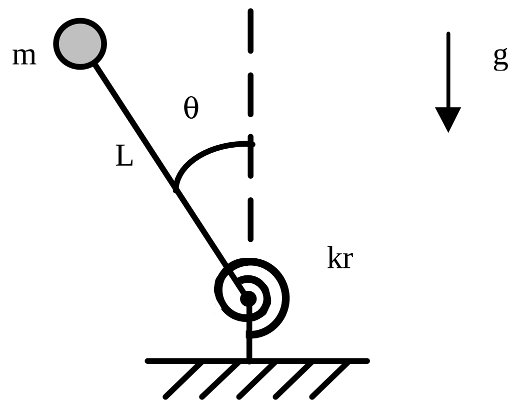


Figure B-1.

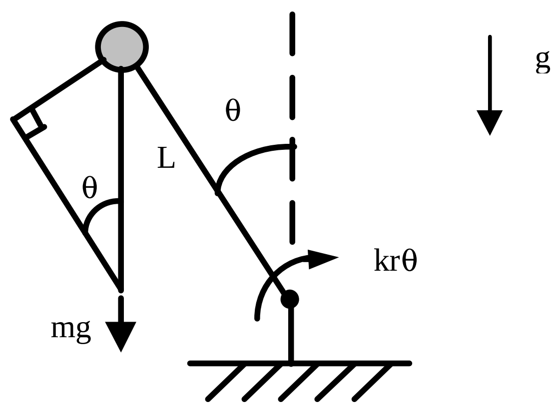


Figure B-2.

kr is the rotational stiffness.

Sum the moments about the pivot point. Let counter-clockwise be positive.

$$\sum M = mL^2\,\ddot{\theta} \tag{B-1}$$

$$m\,L^2\,\ddot{\theta} = mgL\sin\theta - kr\,\theta \tag{B-2}$$

$$m L^2 \ddot{\theta} - mgL \sin\theta + kr\,\theta = 0 \tag{B-3}$$

The resulting equation is nonlinear.

$$\ddot{\theta} - \left[\frac{g}{L}\right]\sin\theta + \left[\frac{kr}{mL^2}\right]\theta = 0 \tag{B-4}$$

Now assume small angular displacements, such that

$$\sin\theta \approx \theta \tag{B-5}$$

By substitution,

$$\ddot{\theta} + \left\{-\left[\frac{g}{L}\right] + \left[\frac{kr}{mL^2}\right]\right\}\theta = 0 \tag{B-6}$$

The natural frequency is thus

$$\omega_n = \sqrt{\left\{-\left[\frac{g}{L}\right] + \left[\frac{kr}{mL^2}\right]\right\}} \tag{B-7}$$

The stability requirement for a small displacement is

$$\frac{kr}{mL^2} > \left[\frac{g}{L}\right] \tag{B-8}$$

$$kr > mgL \tag{B-9}$$

Simple Pendulum Subjected to Base Excitation

Consider the system in Figure C-1. Define a coordinate system at the pendulum rest position in Figure C-2. Note that coordinate system origin is fixed within the enclosure

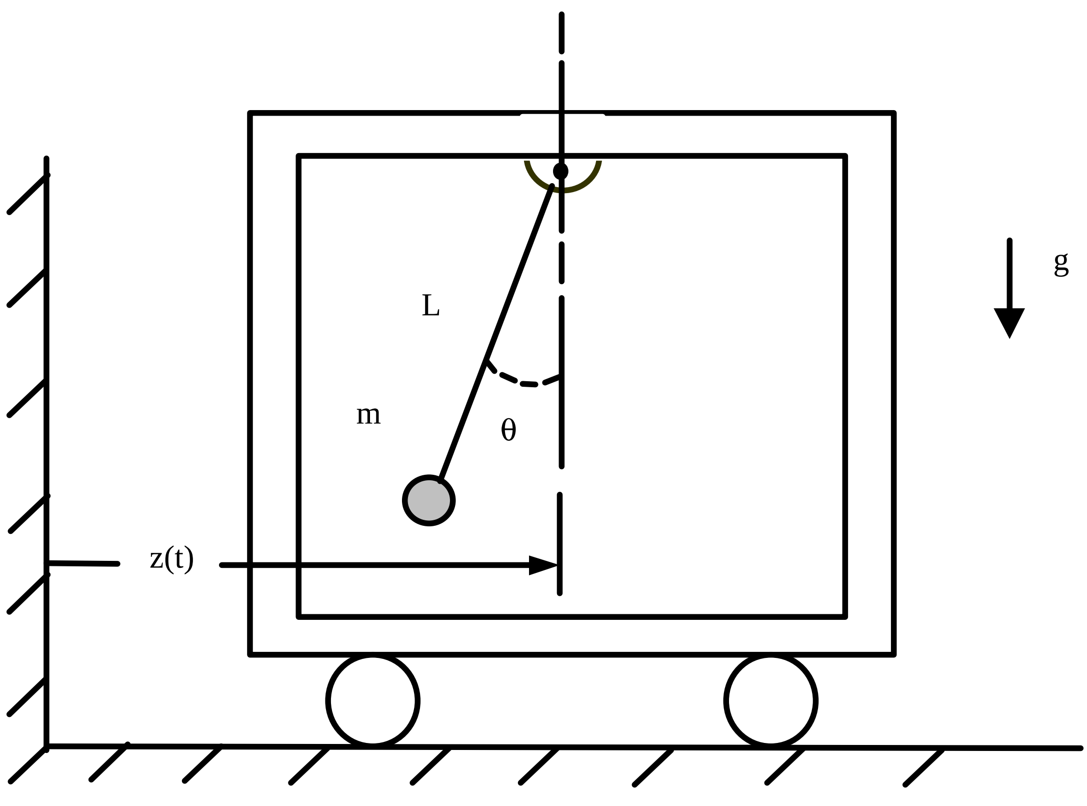


Figure C-1.

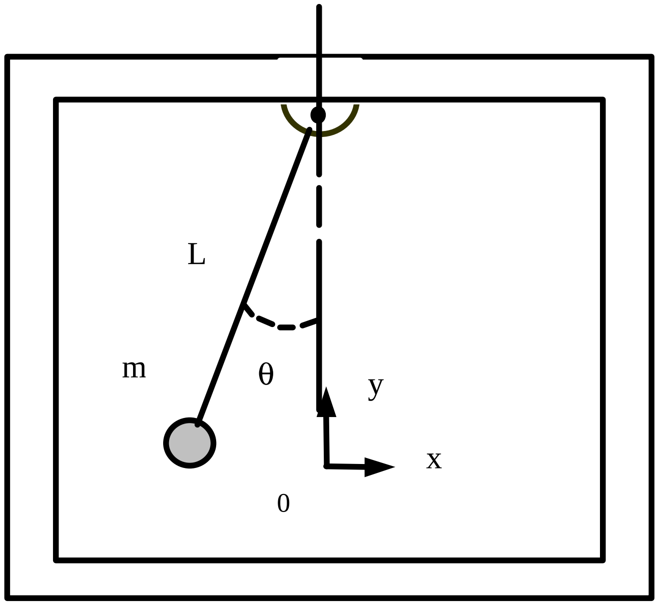


Figure C-2.

Derive the equation of motion. First, determine the position and velocity functions of the pendulum relative to rest position.

$$x = -L \sin\theta \tag{C-1}$$

$$\dot{x} = -\dot{\theta} L \cos\theta \tag{C-2}$$

$$y = L(1 - \cos\theta) \tag{C-3}$$

$$\dot{y} = \dot{\theta} L \sin\theta \tag{C-4}$$

The kinetic energy T is

$$T = \frac{1}{2} m \left\{ \left[\dot{z} - \left(\dot{\theta} L \right) \cos\theta \right]^2 + \left[\left(\dot{\theta} L \right) \sin\theta \right]^2 \right\} \tag{C-5}$$

$$T = \frac{1}{2} m \left\{ \dot{z}^2 - 2\left(\dot{z}\, \dot{\theta} L \right) \cos\theta + \left(\dot{\theta} L \right)^2 \cos^2\theta + \left(\dot{\theta} L \right)^2 \sin^2\theta \right\} \tag{C-6}$$

$$T = \frac{1}{2} m \left\{ \dot{z}^2 - 2\left(\dot{z}\,\dot{\theta}L\right)\cos\theta + \left(\dot{\theta}L\right)^2 \right\} \tag{C-7}$$

The potential energy V is

$$V = m\,g\,L\left[1 - \cos\theta\right] \tag{C-8}$$

Lagrange's equation for a system with a nonconservative force is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i\,, \quad i = 1, 2, \cdots, N \tag{C-9}$$

Apply Lagrange's equation to coordinate θ.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2} m \left\{ \dot{z}^2 - 2\left(\dot{z}\,\dot{\theta}L\right)\cos\theta + \left(\dot{\theta}L\right)^2 \right\}\right]\right\} \tag{C-15}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m\frac{d}{dt}\left\{-\left(\dot{z}\,L\right)\cos\theta + \left(\dot{\theta}L^2\right)\right\} \tag{C-16}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m\left\{-\left(\ddot{z}\,L\right)\cos\theta + \left(\dot{z}\,\dot{\theta}\,L\right)\sin\theta + \left(\ddot{\theta}L^2\right)\right\} \tag{C-17}$$

$$\frac{\partial T}{\partial \theta} = \frac{\partial}{\partial \theta}\left\{\frac{1}{2} m \left\{ \dot{z}^2 - 2\left(\dot{z}\,\dot{\theta}L\right)\cos\theta + \left(\dot{\theta}L\right)^2 \right\}\right\} \tag{C-18}$$

$$\frac{\partial T}{\partial \theta} = m\left(\dot{z}\,\dot{\theta}L\right)\sin\theta \tag{C-19}$$

$$\frac{\partial U}{\partial \theta} = \frac{\partial}{\partial \theta}\left\{m\,g\,L\left[1 - \cos\theta\right]\right\} \tag{C-20}$$

$$\frac{\partial U}{\partial \theta} = \left\{m\,g\,L\sin\theta\right\} \tag{C-21}$$

Lagrange's equation for the θ coordinate is thus

$$m\left\{-(\ddot{z}L)\cos\theta+\left(\dot{z}\dot{\theta}L\right)\sin\theta+\left(\ddot{\theta}L^2\right)\right\}-m\left(\dot{z}\dot{\theta}L\right)\sin\theta+m\,g\,L\sin\theta=0 \qquad \text{(C-24)}$$

$$\left\{-\ddot{z}\cos\theta+\left(\dot{z}\dot{\theta}\right)\sin\theta+\left(\ddot{\theta}L\right)\right\}-\left(\dot{z}\dot{\theta}\right)\sin\theta+g\;\sin\theta=0 \qquad \text{(C-25)}$$

$$-\ddot{z}\cos\theta+\left(\ddot{\theta}L\right)+g\sin\theta=0 \qquad \text{(C-26)}$$

$$\left(\ddot{\theta}L\right)+g\sin\theta=\ddot{z}\cos\theta \qquad \text{(C-27)}$$

For small angular displacement, the equation simplifies to

$$\ddot{\theta}L+g\,\theta=\ddot{z} \qquad \text{(C-28)}$$

Now return to equation (27). Assume that the enclosure is subjected to a constant acceleration

$$\ddot{z}=a \qquad \text{(C-29)}$$

By substitution,

$$\left(\ddot{\theta}L\right)+g\sin\theta=a\cos\theta \qquad \text{(C-30)}$$

A new equilibrium condition is obtained for a constant base excitation. The steady-state condition for a constant base excitation is

$$\ddot{\theta}=0 \qquad \text{(C-31)}$$

$$\theta=\theta_o \qquad \text{(C-32)}$$

By substitution,

$$g\sin\theta_o=a\cos\theta_o \qquad \text{(C-33)}$$

$$g\tan\theta_o=a \qquad \text{(C-34)}$$

Equation (C-34) is can be used to calibrate a simple pendulum accelerometer. Such accelerometers can be used to measure the steady-state acceleration of an automobile or a roller coaster.