NATURAL FREQUENCIES OF BEAMS SUBJECTED TO A UNIFORM AXIAL LOAD Revision A

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Introduction

The natural frequency of a beam is increased by an axial tension load and decreased by an axial compressive load.

The governing differential equation for the transverse displacement y(x, t) is

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{d^2}{\partial x^2}y(x,t)\right\}+P_o\,\frac{\partial}{\partial x}\left[\left(1-\frac{x}{L}\right)\frac{\partial y}{\partial x}\right]+\ m\ \frac{\partial^2\ y(x,t)}{\partial t^2}\ =\ 0 \tag{1}$$

where

E	is the modulus of elasticity
I	is the area moment of inertia
m	is the mass per length
L	is the length
P_o	is the axial compressive load

Equation (1) is taken from Reference 1.

Natural Frequency Formulas

Natural frequency formulas are given in References 2 through 4.

The exact natural frequency f_n for a pinned-pinned or sliding-sliding beam is

$$f_n = \frac{n^2 \pi^2}{2\pi L^2} \sqrt{1 + \frac{PL^2}{EI\, n^2 \pi^2}} \sqrt{\frac{EI}{m}} \quad , \qquad n = 1, 2, 3, \ldots \tag{2}$$

Note that P is positive for a tension load. P is negative for a compression load.

The exact natural frequency for a sliding-pinned beam is

$$f_n = \frac{n^2 \pi^2}{8\pi L^2} \sqrt{1 + \frac{4PL^2}{EI\, n^2 \pi^2}} \sqrt{\frac{EI}{m}} \quad , \qquad n = 1, 3, 5, \ldots \tag{3}$$

The approximate natural frequency formula for beams with other boundary conditions is

$$\frac{f_n \big|_{P \neq 0}}{f_n \big|_{P=0}} = \sqrt{1 + \frac{P}{\left| P_{cr} \right|} \frac{\lambda_1^2}{\lambda_n^2}} \quad , \qquad n = 1, 2, 3, \ldots \tag{4}$$

P_{cr} is the buckling load, as given in Appendix A for common boundary conditions.

λ_n is the non-dimensional frequency in the absence of an axial load. Values for common boundary conditions are given in Appendix B.

Note that the fundamental frequency approaches zero as P approaches the negative critical load. Again, the negative sign corresponds to compression.

An example is shown in Appendix C.

References

1. L. Meirovitch, Analytical Methods in Vibration, Macmillan, New York, 1967.
2. R. Blevins, Formulas for Natural Frequency and Mode Shape, Krieger, Malabar, Florida, 1979.
3. Shaker, F.J., “Effect of Axial Load on Mode Shapes and Frequencies of Beams,” Lewis Research Center Report NASA-TN-8109, December 1975.
4. C. Harris, editor; Shock and Vibration Handbook, 4th edition; W. Stokey, "Vibration of Systems Having Distributed Mass and Elasticity," McGraw-Hill, New York, 1988.
5. Timoshenko and Gere, Theory of Elastic Stability, International Student Edition, 2nd Edition, McGraw-Hill, New Delhi, 1963.
6. Alexander Chajes, Principles of Structural Stability Theory, Prentice-Hall, New Jersey, 1974.
7. T. Irvine, Application of the Newton-Raphson Method to Vibration Problems, Revision B, Vibrationdata Publications, 1999.
8. T. Irvine, Bending Frequencies of Beams, Rods, and Pipes, Revision H; Vibrationdata Publications, 2002.

APPENDIX A

Critical Buckling Loads for Beams with a Constant Axial Load

Boundary Condition	P_{cr}
Fixed-Fixed	$\frac{4\pi^2 EI}{L^2}$
Fixed-Pinned	$\frac{20.19\pi^2 EI}{L^2}$
Fixed-Free	$\frac{\pi^2 EI}{4L^2}$
Pinned-Pinned or Free-Free	$\frac{\pi^2 EI}{L^2}$

The critical loads are taken from References 5 and 6.

APPENDIX B

Non-dimensional Frequency Parameters

The values in the following tables are taken from Reference 7.

Table B-1. Fixed-Free	
n	λ_n
1	1.875104
2	4.694091

Table B-2. Free-Free or Fixed-Fixed	
n	λ_n
1	4.73004
2	7.85320

Table B-3. Free-Pinned or Fixed-Pinned	
n	λ_n
1	3.926602
2	7.068583

APPENDIX C

Example

Consider a fixed-free beam made from aluminum. The beam is 24 inches long. It has a circular cross-section with a diameter of 1 inch. It is subjected to an axial load of +833 lbf, where the positive sign indicates tension. Calculate the fundamental frequency for the loaded beam.

E	=	9900000. lbf/in^2
I	=	0.0490874 in^4
m	=	0.098 lbm/in
L	=	24 inch
P	=	833 lbf

The fundamental frequency for the beam with no axial load is

$$f_1 = \frac{1}{2\pi}\left[\frac{3.5156}{L^2}\right]\sqrt{\frac{EI}{m}} \tag{C-1}$$

$$f_1 = 47.8 \text{ Hz} \quad (\text{ for } P = 0) \tag{C-2}$$

Equation (C-1) is taken from Reference 8.

The critical buckling load is

$$Pcr = \frac{\pi^2 EI}{4L^2} \tag{C-3}$$

$$Pcr = 2082 \text{ lbf} \tag{C-4}$$

$$\frac{f_n \big|_{P \neq 0}}{f_n \big|_{P=0}} = \sqrt{1 + \frac{P}{\left| P_{cr} \right|} \frac{\lambda_1^2}{\lambda_n^2}} \quad , \qquad n = 1, 2, 3, \ldots \tag{C-5}$$

The fundamental frequency requires n = 1.

$$\frac{f_1 \big|_{P \neq 0}}{f_1 \big|_{P=0}} = \sqrt{1 + \frac{P}{\left| P_{cr} \right|}} \tag{C-6}$$

$$\frac{f_1 \big|_{P \neq 0}}{f_1 \big|_{P=0}} = \sqrt{1 + \frac{833}{\left| 2082 \right|}} \tag{C-7}$$

$$\frac{f_1 \big|_{P \neq 0}}{f_1 \big|_{P=0}} = 1.18 \tag{C-8}$$

$$f_1 \big|_{P \neq 0} = 1.18 \left(47.8 \text{ Hz} \right) \tag{C-9}$$

$$f_1 \big|_{P \neq 0} = 56.6 \text{ Hz} \tag{C-10}$$

The result was verified using a finite element model. The software was NE/Nastran. The finite element frequency was 55.87 Hz, using a model with 24 elements. The mode shape is shown in Figure C-1. The input file is shown in Appendix D.

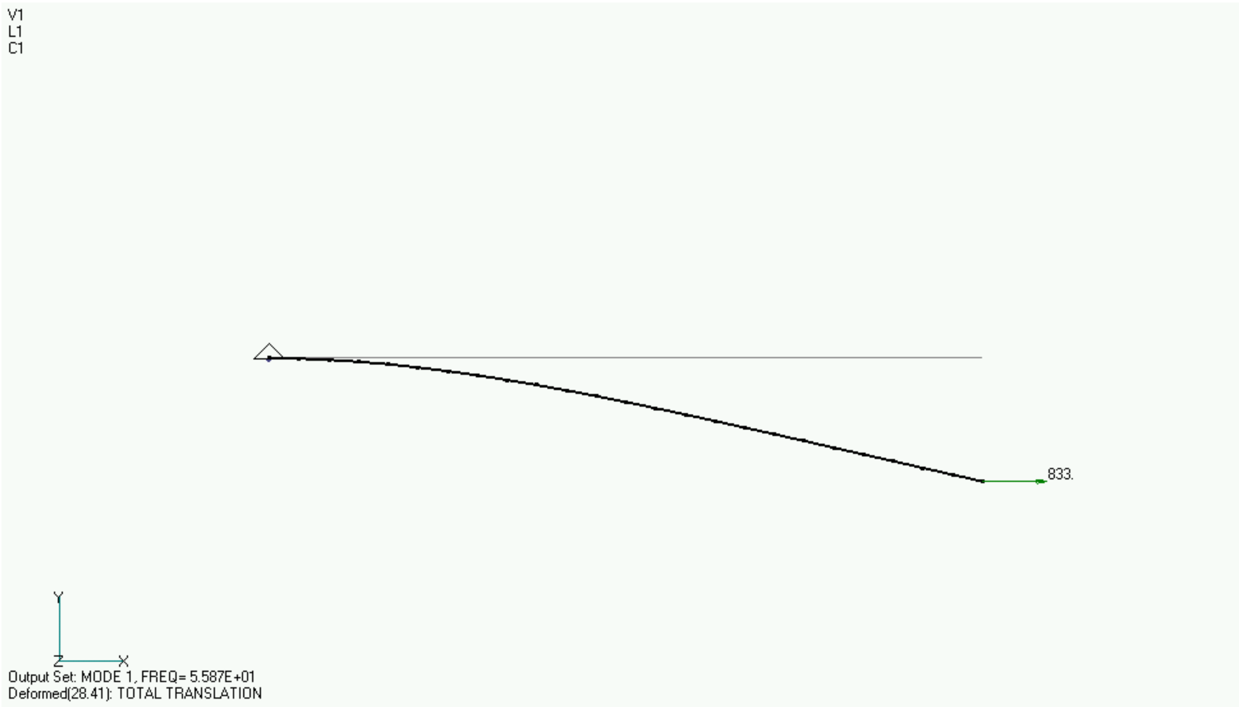

Figure C-1. Fundamental Mode Shape, Cantilever Beam with Axial Load

APPENDIX D

```
ID C:\NENASTRAN82\Modeler\ca,NE/Na
SOL LINEAR PRESTRESS MODAL
TIME 10000
CEND
  TITLE = Normal Modes
  ECHO = NONE
  DISPLACEMENT(PRINT) = ALL
SUBCASE 1
  LABEL = PRESTRESS TENSILE LOAD (AXIAL)
  SPC = 1
  LOAD = 1
SUBCASE 2
  LABEL = MODAL
  SPC = 1
  METHOD = 1
BEGIN BULK
$
******************************************************************
*********
$   Written by : NE/Nastran for Windows
$   Version    : 8.20
$   Translator : NE/Nastran
$   From Model : C:\NENASTRAN82\Modeler\cantilever_beam3.MOD
$   Date       : Mon Jul 21 12:05:50 2003
$
******************************************************************
*********
$
PARAM,POST,-1
PARAM,OGEOM,NO
PARAM,AUTOSPC,YES
PARAM,GRDPNT,0
EIGRL          1                     10
MASS
CORD2C         1       0      0.      0.      0.      0.      0.
1.+NE/NAC1
+NE/NAC1      1.      0.      1.
CORD2S         2       0      0.      0.      0.      0.      0.
1.+NE/NAC2
+NE/NAC2      1.      0.      1.
$ NE/Nastran for Windows Load Set 1 : set 1
FORCE          1      25       0      1.    833.      0.      0.
$ NE/Nastran for Windows Constraint Set 1 : set 1
SPC            1       1  123456      0.
$ NE/Nastran for Windows Property 1 : Bar
PBAR           1       1 0.785070.0490870.0490870.098092      0.
+PR    1
+PR    1      0.    -0.5     0.5      0.      0.     0.5    -0.5
0.+PA    1
+PA    1 0.88652 0.88653      0.
$ NE/Nastran for Windows Material 1 : 6061-T651 Al Plate .25-2.
MAT1           19900000.           0.332.539E-41.265E-5     70.
+MT    1
+MT    1  35000.  35000.  27000.
```

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MAT4           12.060E-3  81.1442.539E-4
GRID           1       0      0.      0.      0.       0
GRID           2       0      1.      0.      0.       0
GRID           3       0      2.      0.      0.       0
GRID           4       0      3.      0.      0.       0
GRID           5       0      4.      0.      0.       0
GRID           6       0      5.      0.      0.       0
GRID           7       0      6.      0.      0.       0
GRID           8       0      7.      0.      0.       0
GRID           9       0      8.      0.      0.       0
GRID          10       0      9.      0.      0.       0
GRID          11       0     10.      0.      0.       0
GRID          12       0     11.      0.      0.       0
GRID          13       0     12.      0.      0.       0
GRID          14       0     13.      0.      0.       0
GRID          15       0     14.      0.      0.       0
GRID          16       0     15.      0.      0.       0
GRID          17       0     16.      0.      0.       0
GRID          18       0     17.      0.      0.       0
GRID          19       0     18.      0.      0.       0
GRID          20       0     19.      0.      0.       0
GRID          21       0     20.      0.      0.       0
GRID          22       0     21.      0.      0.       0
GRID          23       0     22.      0.      0.       0
GRID          24       0     23.      0.      0.       0
GRID          25       0     24.      0.      0.       0
CBAR           1       1       1       2      0.      1.      0.
CBAR           2       1       2       3      0.      1.      0.
CBAR           3       1       3       4      0.      1.      0.
CBAR           4       1       4       5      0.      1.      0.
CBAR           5       1       5       6      0.      1.      0.
CBAR           6       1       6       7      0.      1.      0.
CBAR           7       1       7       8      0.      1.      0.
CBAR           8       1       8       9      0.      1.      0.
CBAR           9       1       9      10      0.      1.      0.
CBAR          10       1      10      11      0.      1.      0.
CBAR          11       1      11      12      0.      1.      0.
CBAR          12       1      12      13      0.      1.      0.
CBAR          13       1      13      14      0.      1.      0.
CBAR          14       1      14      15      0.      1.      0.
CBAR          15       1      15      16      0.      1.      0.
CBAR          16       1      16      17      0.      1.      0.
CBAR          17       1      17      18      0.      1.      0.
CBAR          18       1      18      19      0.      1.      0.
CBAR          19       1      19      20      0.      1.      0.
CBAR          20       1      20      21      0.      1.      0.
CBAR          21       1      21      22      0.      1.      0.
CBAR          22       1      22      23      0.      1.      0.
CBAR          23       1      23      24      0.      1.      0.
CBAR          24       1      24      25      0.      1.      0.
ENDDATA
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