AN INTRODUCTION TO FREQUENCY RESPONSE FUNCTIONS

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Introduction

The purpose of this report is to discuss frequency response functions. These functions are used in vibration analysis and modal testing. The purpose of modal testing is to identify the natural frequencies, damping ratios, and mode shapes of a structure.

Natural Frequencies

Bridges, aircraft wings, machine tools, and all other physical structures have natural frequencies. A natural frequency is the frequency at which the structure would oscillate if it were disturbed from its rest position and then allowed to vibrate freely. All structures have at least one natural frequency. Nearly every structure has multiple natural frequencies.

Resonance occurs when the applied force or base excitation frequency coincides with a structural natural frequency. During resonant vibration, the response displacement may increase until the structure experiences buckling, yielding, fatigue, or some other failure mechanism.

The failure of the Cypress Viaduct in the 1989 Loma Prieta Earthquake is example of failure due to resonant excitation. Resonant vibration caused 50 of the 124 spans of the Viaduct to collapse. The reinforced concrete frames of those spans were mounted on weak soil. As a result, the natural frequency of those spans coincided with the frequency of the earthquake ground motion. The Viaduct structure thus amplified the ground motion. The spans suffered increasing vertical motion. Cracks formed in the support frames. Finally, the upper roadway collapsed, slamming down on the lower road.

Dynamic Analysis

Engineers performing dynamic analysis must

1. Determine the natural frequencies of the structure.
2. Characterize potential excitation functions.
3. Calculate the response of the structure to the maximum expected excitation.
4. Determine whether the expected response violates any failure criteria.

This report is concerned with the first step.

The natural frequencies can be calculated via analytical methods during the design stage. The frequencies may also be measured after the structure, or a prototype, is built.

Each natural frequency has a corresponding damping ratio. Damping values are empirical values that must be obtained by measurement.

Frequency Response Function Overview

There are many tools available for performing vibration analysis and testing. The frequency response function is a particular tool.

A frequency response function (FRF) is a transfer function, expressed in the frequency-domain.

Frequency response functions are complex functions, with real and imaginary components. They may also be represented in terms of magnitude and phase.

A frequency response function can be formed from either measured data or analytical functions.

A frequency response function expresses the structural response to an applied force as a function of frequency. The response may be given in terms of displacement, velocity, or acceleration. Furthermore, the response parameter may appear in the numerator or denominator of the transfer function.

Frequency Response Function Model

Consider a linear system as represented by the diagram in Figure 1.

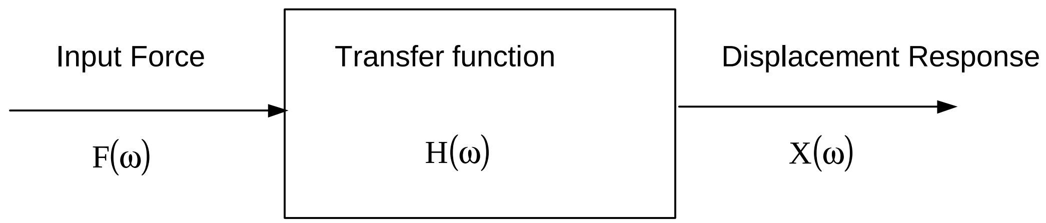


Figure 1.

$F(\omega)$ is the input force as a function of the angular frequency ω. $H(\omega)$ is the transfer function. $X(\omega)$ is the displacement response function. Each function is a complex function, which may also be represented in terms of magnitude and phase. Each function is thus a spectral function. There are numerous types of spectral functions. For simplicity, consider each to be a Fourier transform.

The relationship in Figure 1 can be represented by the following equations

$$X(\omega) = H(\omega) \cdot F(\omega) \tag{1}$$

$$H(\omega) = \frac{X(\omega)}{F(\omega)} \tag{2}$$

Similar transfer functions can be developed for the velocity and acceleration responses.

<u>Nomenclature</u>

There are six basic transfer functions as shown in Tables 1 and 2.

Table 1. Frequency Response Function Names			
Dimension	Displacement / Force	Velocity / Force	Acceleration / Force
Name	Admittance, Compliance, Receptance	Mobility	Accelerance, Inertance

Table 2. Frequency Response Function Names			
Dimension	Force / Displacement	Force / Velocity	Force / Acceleration
Name	Dynamic Stiffness	Mechanical Impedance	Apparent Mass, Dynamic Mass

Note that all of the functions in Tables 1 and 2 are related by algebraic equations. Any of the function can be calculated from any other.

<u>Analytical Frequency Response Function</u>

Consider a single-degree-of-freedom system subjected to a force excitation as shown in Figure 2. The free-body diagram is shown in Figure 3.

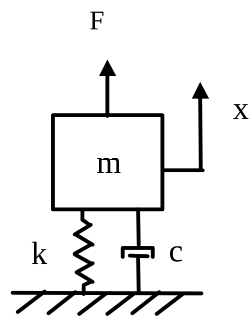


Figure 2. Single-degree-of-freedom System

The variables are

m = mass,
c = viscous damping coefficient,
k = stiffness,
x = absolute displacement of the mass,
F = applied force.

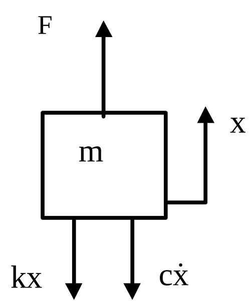


Figure 3. Free-body Diagram

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{3}$$

$$m\ddot{x} = -c\dot{x} - kx + F \tag{4}$$

$$m\ddot{x} + c\dot{x} + kx = F \tag{5}$$

$$\ddot{x} + (c/m)\dot{x} + (k/m)x = F/m \tag{6}$$

By convention,

$$(c/m) = 2\xi\omega_n \tag{7}$$

$$(k/m) = \omega_n^{\,2} \tag{8}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substituting the convention terms into equation (6),

$$\ddot{x} + 2\xi\omega_n\,\dot{x} + \omega_n^{\,2}\,x = \omega_n^{\,2}\,F/k \tag{9}$$

The Fourier transform of each side of equation (9) may be taken to derive the steady-state transfer function for the absolute response displacement, as shown in Reference 1.

After many steps, the resulting transfer function is

$$\frac{X(\omega)}{F(\omega)} = \left[\frac{1}{k}\right]\left[\frac{\omega_n^{\,2}}{\omega_n^{\,2} - \omega^2 + j(2\xi\,\omega\omega_n)}\right] \tag{10}$$

This transfer function, which represents displacement over force, is sometimes called the receptance function, as shown in Table 1.

The transfer function can be represented in terms of magnitude and phase angle ϕ as

$$\left|\frac{X(\omega)}{F(\omega)}\right| = \left[\frac{1}{k}\right]\left[\frac{\omega_n^{\,2}}{\sqrt{\left(\omega_n^{\,2} - \omega^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2}}\right] \tag{11a}$$

$$\left|\frac{X(\omega)}{F(\omega)}\right| = \left[\frac{1}{m}\right]\left[\frac{1}{\sqrt{\left(\omega_n^{\,2} - \omega^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2}}\right] \tag{11b}$$

$$\phi = \arctan\left[\frac{2\xi\,\omega\omega_n}{\omega_n^{\,2} - \omega^2}\right] \tag{12}$$

The mobility function is

$$\frac{V(\omega)}{F(\omega)} = \left[\frac{1}{k}\right]\left[\frac{j\,\omega\omega_n{}^2}{\omega_n{}^2 - \omega^2 + j(2\xi\,\omega\omega_n)}\right] \tag{13}$$

$$\left|\frac{V(\omega)}{F(\omega)}\right| = \left[\frac{1}{k}\right]\left[\frac{\omega\omega_n{}^2}{\sqrt{\left(\omega_n{}^2 - \omega^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2}}\right] \tag{14a}$$

$$\left|\frac{V(\omega)}{F(\omega)}\right| = \left[\frac{1}{m}\right]\left[\frac{\omega}{\sqrt{\left(\omega_n{}^2 - \omega^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2}}\right] \tag{14b}$$

$$\theta = \arctan\left[\frac{-\omega_n{}^2 + \omega^2}{2\xi\omega_n}\right] \tag{15}$$

The accelerance function is

$$\frac{A(\omega)}{F(\omega)} = \left[\frac{1}{k}\right]\left[\frac{-\omega^2\,\omega_n{}^2}{\omega_n{}^2 - \omega^2 + j(2\xi\,\omega\omega_n)}\right] \tag{16}$$

$$\left|\frac{A(\omega)}{F(\omega)}\right| = \left[\frac{1}{k}\right]\left[\frac{-\,\omega^2\,\omega_n{}^2}{\sqrt{\left(\omega_n{}^2 - \omega^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2}}\right] \tag{17a}$$

$$\left|\frac{A(\omega)}{F(\omega)}\right| = \left[\frac{1}{m}\right]\left[\frac{-\,\omega^2}{\sqrt{\left(\omega_n{}^2 - \omega^2\right)^2 + \left(2\xi\,\omega\omega_n\right)^2}}\right] \tag{17b}$$

$$\alpha = -\pi + \arctan\left[\frac{2\xi\omega_n}{\omega_n{}^2 - \omega^2}\right] \tag{18}$$

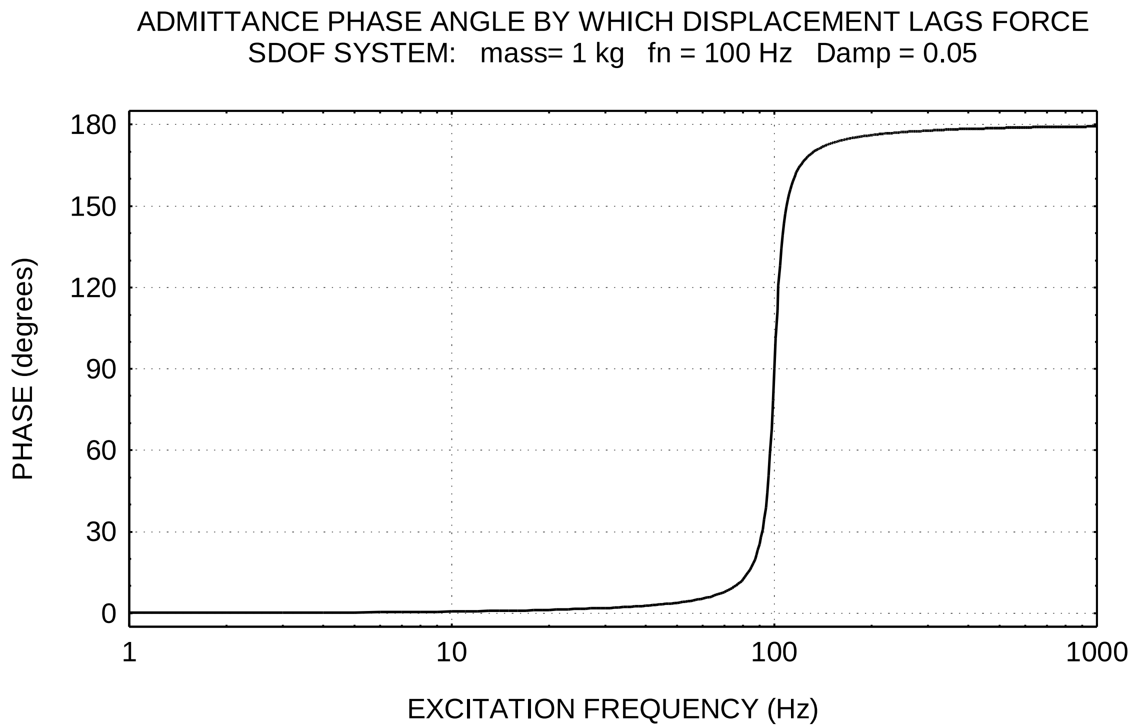
<u>Example</u>

Consider a single-degree-of-freedom system with parameters shown in Table 3.

Table 3. Parameters		
Variable	Symbol	Value
Mass	m	1 kg
Stiffness	k	3.948e+05 N/m
Natural Frequency	ω_n	100 Hz (628.32 rad/sec)
Damping Ratio	ξ	0.05

The systems frequency response functions are plotted according to Table 4. The functions are plotted in terms of amplitude and phase.

Table 4. Plot Format		
Function	Description	Figure
Admittance	Displacement / Force	3
Mobility	Velocity / Force	4
Accelerance	Acceleration / Force	5
Dynamic Stiffness	Force / Displacement	6
Mechanical Impedance	Force / Velocity	7
Apparent Mass	Force / Acceleration	8


ADMITTANCE PHASE ANGLE BY WHICH DISPLACEMENT LAGS FORCE
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
PHASE (degrees)
180
150
120
90
60
30
0
1
10
100
1000
EXCITATION FREQUENCY (Hz)

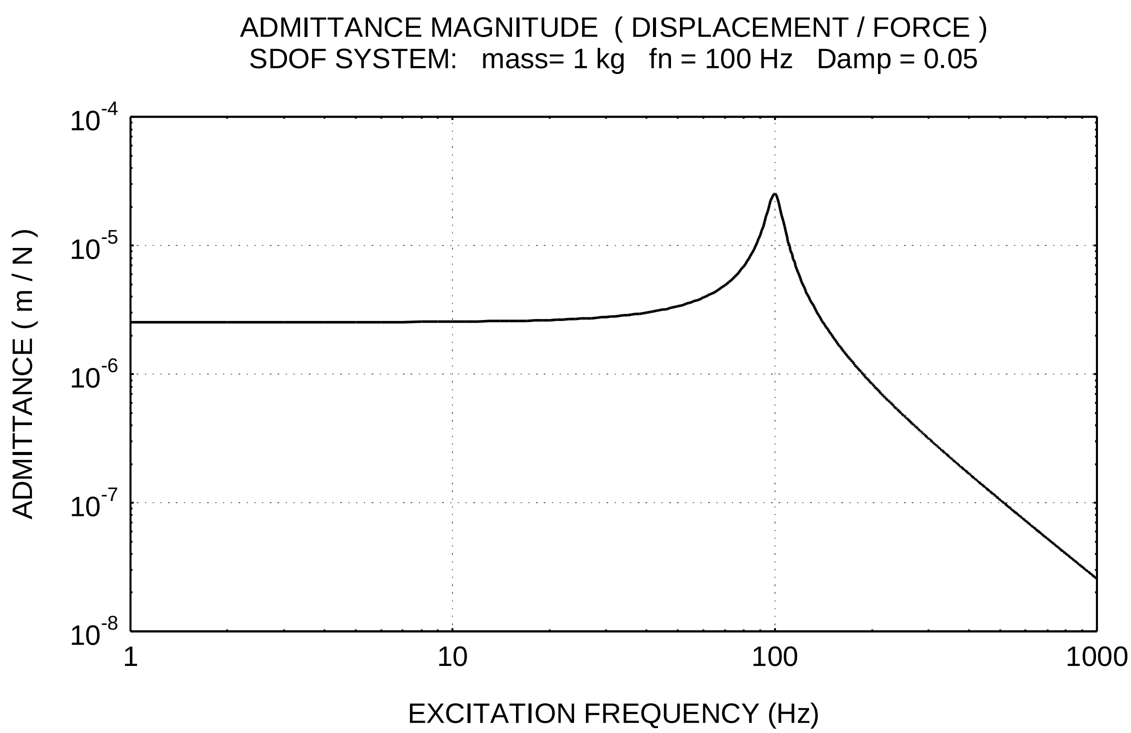
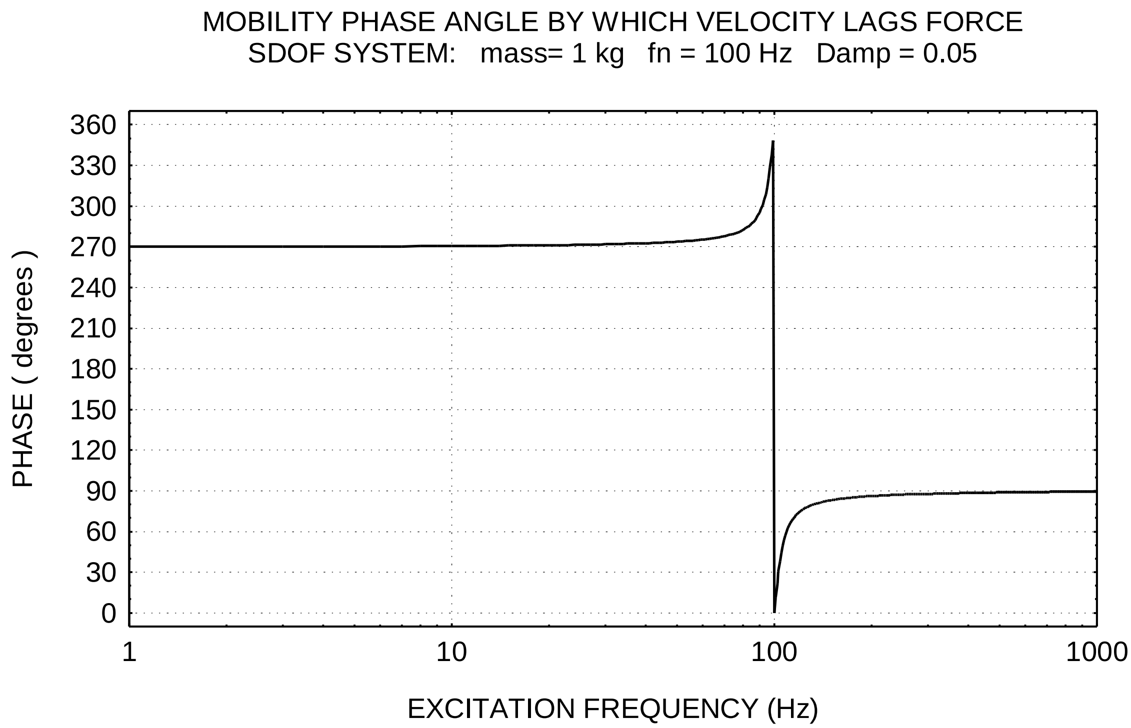

ADMITTANCE MAGNITUDE (DISPLACEMENT / FORCE)
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
ADMITTANCE (m / N)
10^-4
10^-5
10^-6
10^-7
10^-8
1
10
100
1000
EXCITATION FREQUENCY (Hz)

Figure 3.


MOBILITY PHASE ANGLE BY WHICH VELOCITY LAGS FORCE
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
PHASE (degrees)
360
330
300
270
240
210
180
150
120
90
60
30
0
1
10
100
1000
EXCITATION FREQUENCY (Hz)

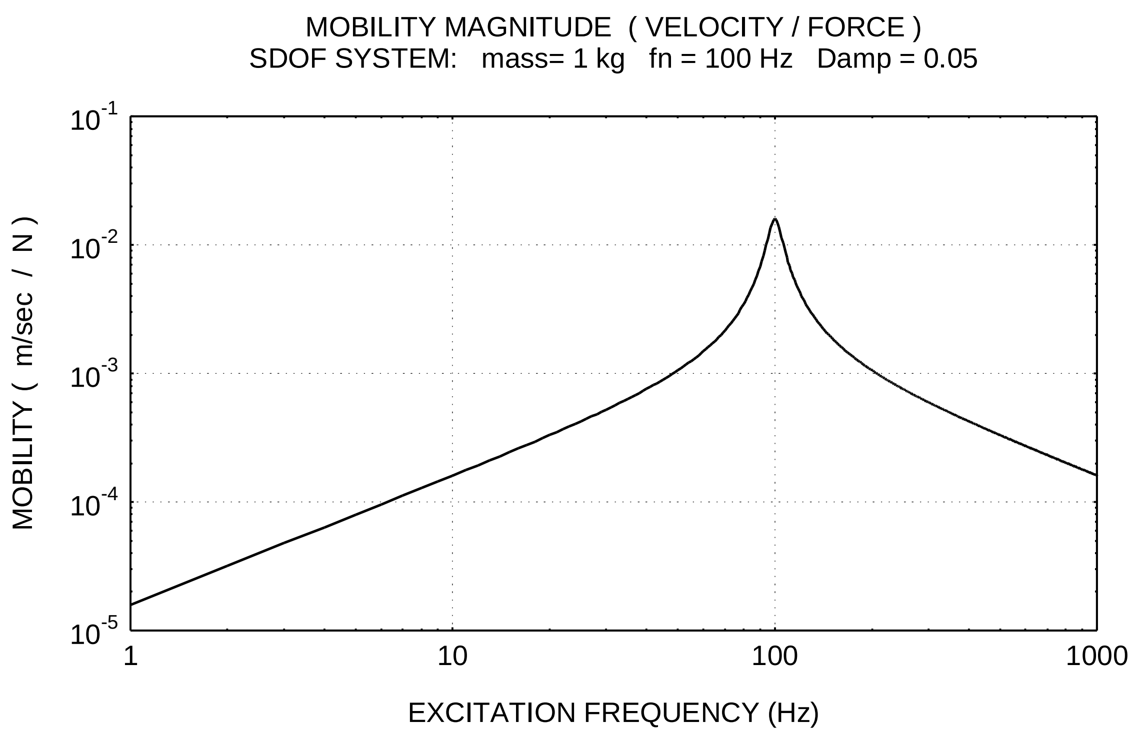
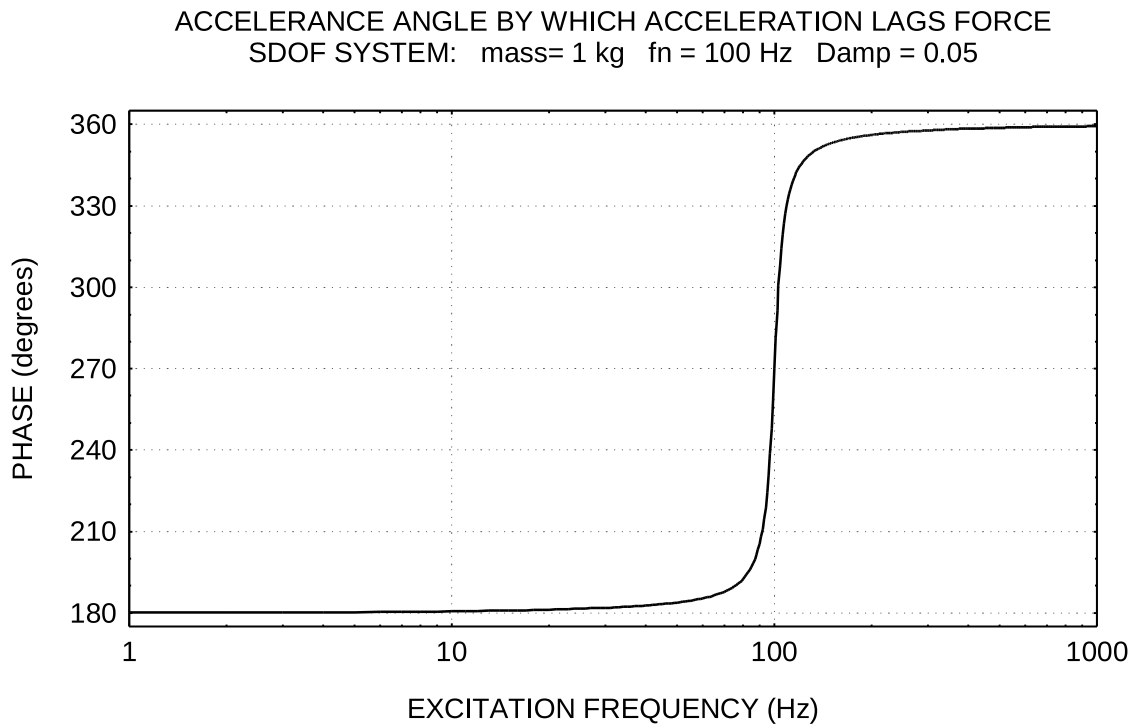

MOBILITY MAGNITUDE (VELOCITY / FORCE)
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
MOBILITY (m/sec / N)
10^-1
10^-2
10^-3
10^-4
10^-5
1
10
100
1000
EXCITATION FREQUENCY (Hz)

Figure 4.


ACCELERANCE ANGLE BY WHICH ACCELERATION LAGS FORCE
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
PHASE (degrees)
360
330
300
270
240
210
180
1
10
100
1000
EXCITATION FREQUENCY (Hz)

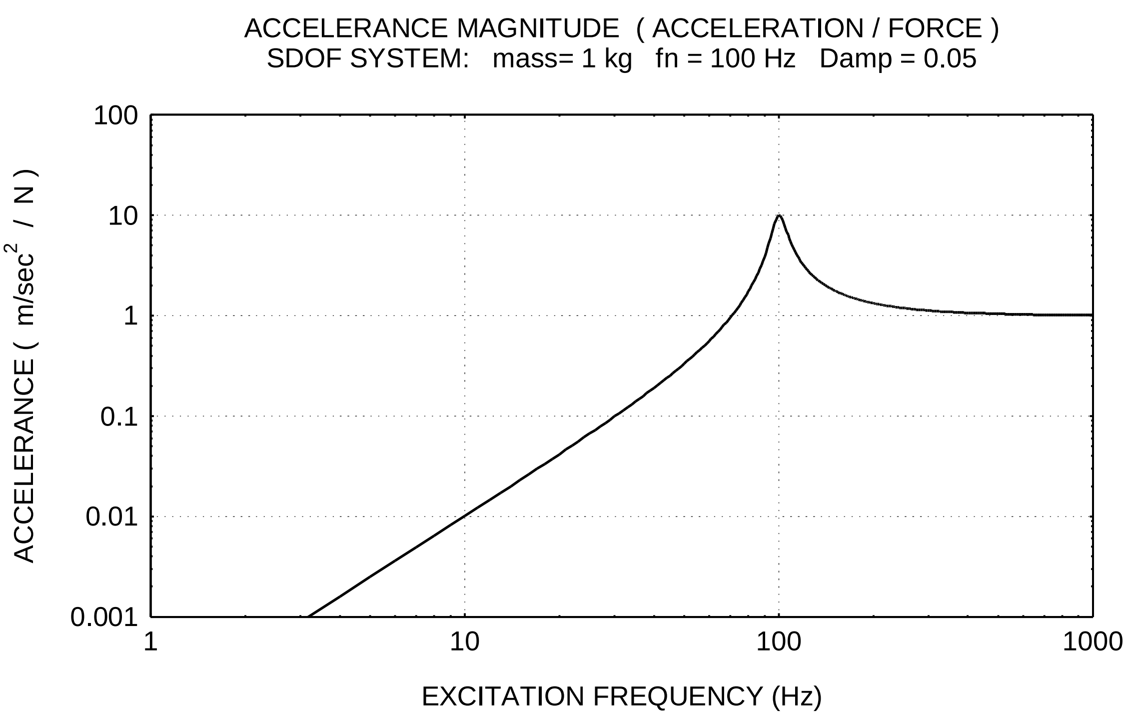

ACCELERANCE MAGNITUDE (ACCELERATION / FORCE)
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
ACCELERANCE (m/sec2 / N)
100
10
1
0.1
0.01
0.001
1
10
100
1000
EXCITATION FREQUENCY (Hz)

Figure 5.

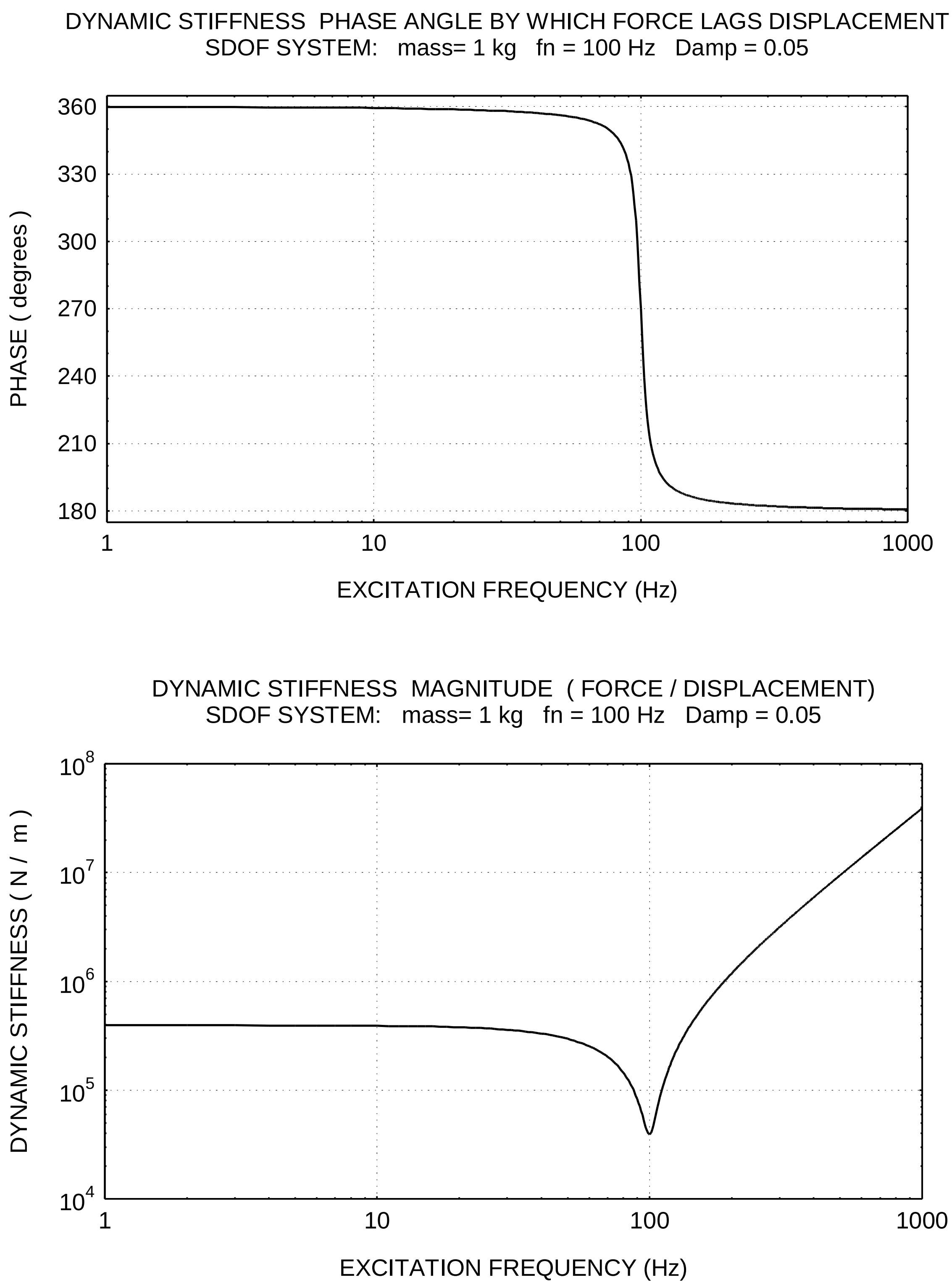
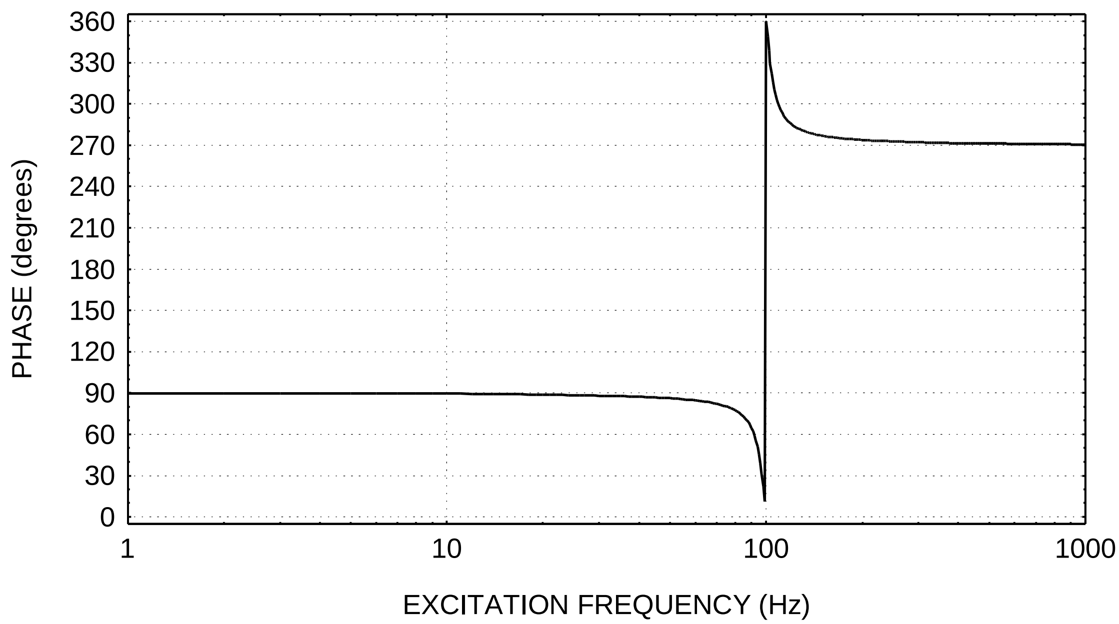

DYNAMIC STIFFNESS PHASE ANGLE BY WHICH FORCE LAGS DISPLACEMENT
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
PHASE (degrees)
360
330
300
270
240
210
180
1
10
100
1000
EXCITATION FREQUENCY (Hz)
DYNAMIC STIFFNESS MAGNITUDE (FORCE / DISPLACEMENT)
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
DYNAMIC STIFFNESS (N / m)
10^8
10^7
10^6
10^5
10^4
1
10
100
1000
EXCITATION FREQUENCY (Hz)

Figure 6.

MECHANICAL IMPEDANCE PHASE ANGLE BY WHICH FORCE LAGS VELOCITY
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05



MECHANICAL IMPEDANCE MAGNITUDE (FORCE / VELOCITY)
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05

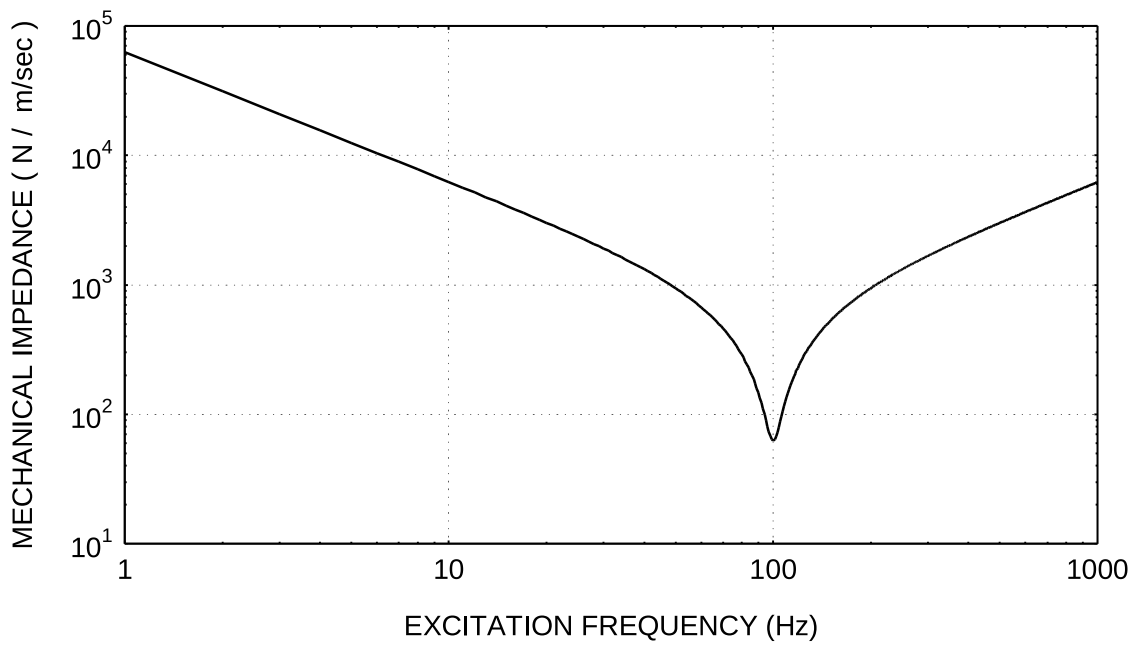
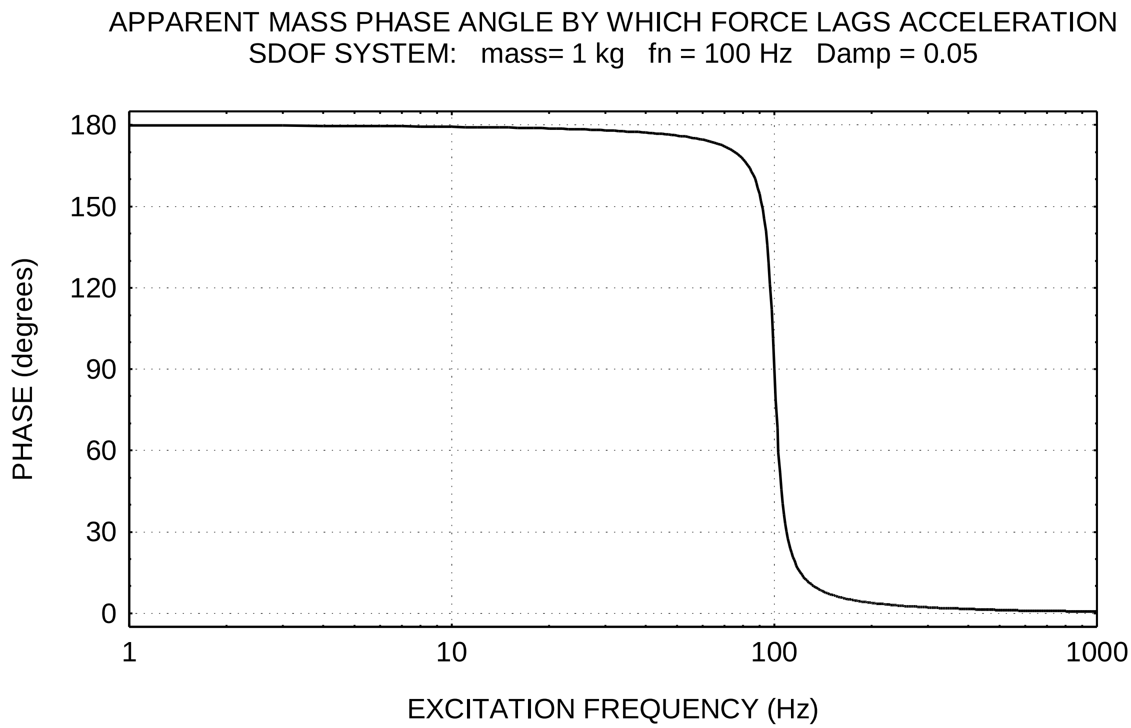


Figure 7.


APPARENT MASS PHASE ANGLE BY WHICH FORCE LAGS ACCELERATION
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
PHASE (degrees)
180
150
120
90
60
30
0
1
10
100
1000
EXCITATION FREQUENCY (Hz)

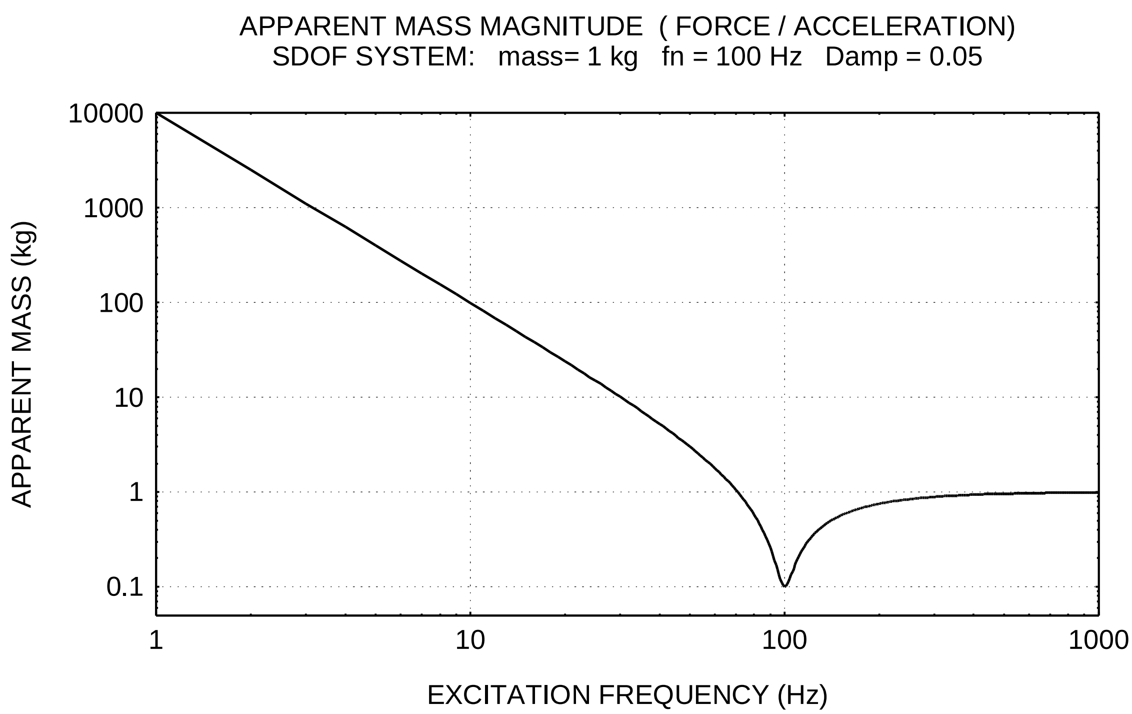

APPARENT MASS MAGNITUDE (FORCE / ACCELERATION)
SDOF SYSTEM: mass= 1 kg fn = 100 Hz Damp = 0.05
APPARENT MASS (kg)
10000
1000
100
10
1
0.1
1
10
100
1000
EXCITATION FREQUENCY (Hz)

Figure 8.

References

1. T. Irvine, The Steady-state Response of a Single-degree-of-freedom System Subjected to a Harmonic Force, Vibrationdata.com Publications, 1999.